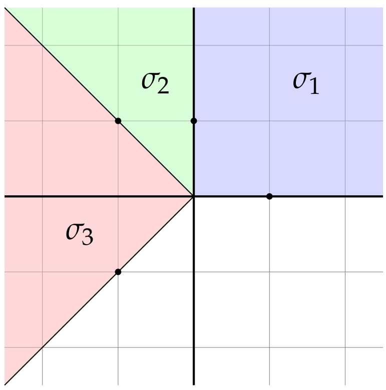
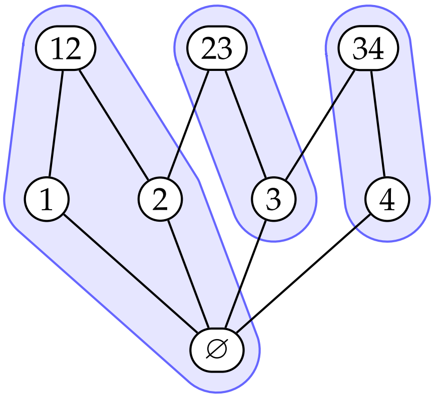
Master Thesis

Chow Rings of Toric Varieties





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Preface

The study of toric varieties, often referred to as toric geometry, goes back to 1970, when the first formal definition of a toric variety by Demazure appeared in [Dem70]. In the following years, toric varieties appeared in publications in different areas, but a work solely dedicated to toric geometry had yet to be written. This happened in 1973 independently in the book [KKMS73] by Kempf, Knudsen, Mumford and Saint-Donat and in the article [MO75] by Oda and Miyake, presented in 1973 but not published until 1975. In the introduction to [KKMS73], Mumford already notices that toric varieties provide a very useful source of examples of algebraic varieties. Apart from applications of toric varieties in recent theoretical physics (see for example [CK99]), this is a key importance of toric varieties. They provide examples of algebraic varieties that allow concrete computations due to their combinatorial nature and therefore serve as a testing ground for theories. The research on toric varieties peaked in 1978, when Danilov published [Dan78], which is a detailed introduction to toric geometry that surveys the results made by others, but also introduces a lot of new ideas to the topic. Another important work appeared in 1993, when Fulton condensed his lectures on toric varieties into the book [Ful93]. In 2011, a very modern treatment of toric varieties was published by Cox, Little and Schenck in their book [CLS11], which brings the research of 40 years together into a great book, giving a detailed account on the historical development of the field as well.

The main topic of this thesis goes back to [Ehl75], an article by Ehlers on toric varieties from a time before the term "toric variety" has been established. Ehlers uses a mathematical language very unfamiliar from a modern point of view, as he discusses toric varieties as complex manifolds, missing the algebro-geometric aspects. In his article, Ehlers finds a linear basis of the homology groups $H_*(X_\Sigma)$ of certain toric varieties X_Σ. This result reappears in [Dan78] in a more standard language. Danilov proves the linear basis for homology groups of smooth projective toric varieties. By Poincaré duality, this yields a linear basis for the cohomology ring $H^*(X_\Sigma)$, which is in turn isomorphic to the combinatorial Chow ring $\mathcal{R}(\Sigma)$ of the underlying polyhedral fan. This ring, directly associated to the underlying fan, is the object of interest in this thesis. Our goal was to prove the linear basis of $\mathcal{R}(\Sigma)$, obtained using algebro-geometric methods by Danilov, by directly looking at the combinatorial Chow ring. This would yield a new proof of the basis theorem using algebraic combinatorics to discuss the combinatorial Chow ring, instead of algebraic geometry to discuss the homology of the associated variety. A first step in this direction was made in [Ful93], where Fulton decouples a combinatorial condition of the fan from the projectivity of the variety. He noticed that the necessary combinatorial condition holds in the projective case, but projectivity is not necessary to prove the linear basis of the homology groups. To our surprise, this linear basis did not reappear in [CLS11], although other results from the works of Danilov and Fulton on the (co)homology of toric varieties are treated.

In this thesis, we take an algebraic combinatorics point of view and try to reprove and generalize the linear basis theorem for the combinatorial Chow ring $\mathcal{R}(\Sigma)$ of a simplicial polyhedral fan Σ. By noticing that Fulton's combinatorial condition is equivalent to shellability of the associated simplicial complex $\Delta(\Sigma)$, we formulate our propositions in terms of shellable simplicial fans. We give algebraic proofs for the desired linear generating set in all dimensions and its linear independence up to dimension 1. Though we know from Danilov and Fulton that linear independence holds in higher dimensions under additional assumptions, we were not able to find an algebraic proof for this, not involving algebraic geometry of the associated variety.

The thesis is split into three chapters. We start with an introduction to toric geometry in the first chapter. The introduction aims to be self-contained without getting entangled in details of algebraic geometry. Thus, we take a very classical point of view on algebraic geometry, avoiding more abstract concepts like sheaves and schemes. At some points we see the disadvantages of this approach, for example when defining $\operatorname{Spec}(R)$, but the great benefit is, that we can focus on the beautiful combinatorics and convex geometry behind toric varieties. This focus is reflected in our definition of affine toric varieties, that does not even mention the embedded torus and its action on the variety. Nevertheless, we discuss the connection to the classical approach of toric varieties as torus embeddings in several remarks. Throughout the thesis, we provide examples to clarify the abstract concepts and fill gaps where proofs are omitted.

The second chapter is the original work of this thesis. We define the combinatorial Chow ring $\mathcal{R}(\Sigma)$ of a simplicial fan Σ in the most general setting and generalize some properties already mentioned in Ewald's book [Ewa96]. As a sanity check for our general definition of $\mathcal{R}(\Sigma)$ we make a quick digression to study the combinatorial Chow ring of products of simplicial fans, which works out nicely as expected from the connection to the Stanley-Reisner ring of the associated simplicial complex. We go on to define shellability of simplicial fans and prove the linear generators of $\mathcal{R}(\Sigma)$ in the shellable case. We prove the linear independence of the generators up to dimension one and formulate our conjecture for higher dimensions.

In the third and final chapter we come back to toric geometry and provide the algebro-geometric context of our work. We show that Fulton's condition is really shellability and from that extract a linear basis for the homology groups of toric varieties given by complete smooth shellable fans. Using Poincaré duality and a result from [Dan78], we verify our conjecture under the additional assumption of completeness.

All of our examples will take place in $\mathbb{R}^n$, where we have the standard basis $e_1, \dots, e_n$ with dual basis $e_1^*, \dots, e_n^*$ defined by $\langle e_i^*, e_j \rangle = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta and $\langle -, - \rangle$ denotes the dual pairing $(\mathbb{R}^n)^* \times \mathbb{R}^n \to \mathbb{R}$, $(f, v) \mapsto f(v)$. As usual in combinatorics, we abbreviate subsets of $[n] = \{1, 2, \dots, n\}$ by strings of elements, e.g. $13 = \{1, 3\}$, when it is clear from the context that we mean the subset $\{1, 3\}$ and not the number thirteen. All rings in this thesis are commutative rings with 1 and ring homomorphisms map 1 to 1. All $\mathbb{C}$-algebras are associative $\mathbb{C}$-algebras with 1 and $\mathbb{C}$-algebra homomorphisms also map 1 to 1. Homology and cohomology groups are taken to have integral coefficients if not mentioned otherwise.

1 Introduction to Toric Geometry

In this chapter we give an introduction to toric geometry, including the necessary background in algebraic geometry. We will not prove every statement, but instead look at examples where they help to grasp the abstract concepts. For details, we refer to [CLO07] and [CLS11], which are textbooks on algebraic geometry and toric geometry, respectively.

1.1 Affine Varieties

To understand toric varieties in general, we need to consider affine toric varieties first. Afterwards we will be able to glue these affine varieties along certain open subsets to obtain toric varieties. We start with the classical definitions of affine algebraic varieties and their coordinate rings.

Definition 1.1. An *affine variety* $V \subseteq \mathbb{C}^n$ is the zero-locus of finitely many polynomials $f_1, f_2, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$,

$$V = \{\, p \in \mathbb{C}^n \mid f_1(p) = f_2(p) = \cdots = f_s(p) = 0 \,\}.$$

Since $\mathbb{C}[x_1, \ldots, x_n]$ is a Noetherian ring, every ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is finitely generated, so the set of all points $p \in \mathbb{C}^n$ with $f(p) = 0$ for all $f \in I$ is an affine variety $\mathbf{V}(I)$. Conversely, given an affine variety $V \subseteq \mathbb{C}^n$, the polynomials vanishing on V form an ideal $\mathbf{I}(V)$. While for every affine variety V the affine variety $\mathbf{V}(\mathbf{I}(V))$ is always V itself, the ideal $\mathbf{I}(\mathbf{V}(I))$ is different from I in general. The relationship is given by Hilbert's Nullstellensatz.

Theorem 1.2 (Hilbert's Nullstellensatz, [CLO07, Chapter 4, §2, Theorem 6]). *Let I be an ideal in the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$, then $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$.* □

Here $\sqrt{I}$ is the *radical* of I, consisting of all polynomials f, such that $f^k \in I$ for some positive integer k.

Example 1.3. Let $f = x^2 + y^2 - 1 \in \mathbb{C}[x, y, z]$, then $V = \mathbf{V}(f)$ is the affine variety in $\mathbb{C}^3$ consisting of all points $(x, y, z) \in \mathbb{C}^3$ such that $x^2 + y^2 = 1$. Intersecting V with $\mathbb{R}^3$ we obtain the real part of an infinite cylinder as shown in Figure 1.1.

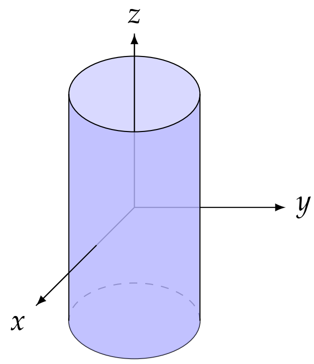


Figure 1.1: The real part of the affine variety $\mathbf{V}(x^2 + y^2 - 1) \subseteq \mathbb{C}^3$.

For $g = (x^2 + y^2 - 1)^3$, we obtain a different ideal $\langle g \rangle \subsetneq \langle f \rangle$, but the same variety $\mathbf{V}(g) = \mathbf{V}(f)$. In fact, $\langle f \rangle$ is radical the radical of $\langle g \rangle$, so $\mathbf{I}(V) = \langle f \rangle$. ◊

The Zariski Topology. In addition to the standard topology on an affine variety $V \subseteq \mathbb{C}^n$ induced by the standard topology on $\mathbb{C}^n$, there is another useful topology on affine varieties. The subvarieties of V (*i.e.*, affine varieties in $\mathbb{C}^n$ that are contained in V) form the collection of closed sets of a topology, called the *Zariski topology* on V. Since subvarieties are also closed in the standard topology, the Zariski topology is coarser than the standard topology. In fact, the Zariski topology is usually not even Hausdorff. Consider $V = \mathbb{C}$, the only subvarieties of V are finite point sets, so the Zariski topology is the cofinite topology in this case, which is not Hausdorff.

Morphisms of Affine Varieties. A map $\phi\colon V \to W$ between affine varieties that is given by polynomials in each coordinate is called a *morphism of affine varieties*. Affine varieties together with morphisms of affine varieties form a category. In particular, we say that two affine varieties V and W are isomorphic if there exist morphisms $\phi\colon V \to W$ and $\psi\colon W \to V$, such that $\psi \circ \phi = \mathrm{id}_V$ and $\phi \circ \psi = \mathrm{id}_W$.

Coordinate Rings of Affine Varieties. To every affine variety V we associate a $\mathbb{C}$-algebra with elements corresponding to morphisms $V \to \mathbb{C}$.

Definition 1.4. Given an affine variety $V \subseteq \mathbb{C}^n$, we define the *coordinate ring of* V to be the $\mathbb{C}$-algebra $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]/\mathbf{I}(V)$.

A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ corresponds to a polynomial map $f\colon \mathbb{C}^n \to \mathbb{C}$, so it gives a morphism $f|_V\colon V \to \mathbb{C}$. For two polynomials f and g we have $f|_V = g|_V$ if and only if $f - g \in \mathbf{I}(V)$. Hence, elements of $\mathbb{C}[V]$ correspond to morphisms $V \to \mathbb{C}$. A different formulation of Hilbert's Nullstellensatz tells us that the points of V are in one to one correspondence with maximal ideals in $\mathbb{C}[V]$, where $p \in V$ corresponds to the ideal consisting of all $f \in \mathbb{C}[V]$ vanishing on p, see [CLO07, Chapter 5, §4, Theorem 5]. Since every morphism $\phi\colon V \to W$ of affine varieties induces a $\mathbb{C}$-algebra homomorphism $\phi^*\colon \mathbb{C}[W] \to \mathbb{C}[V], f \mapsto f \circ \phi$, we have a contravariant functor from the category of affine varieties to the category of $\mathbb{C}$-algebras, that assigns to each affine variety V its coordinate ring $\mathbb{C}[V]$ and to each morphism $\phi\colon V \to W$ the induced homomorphism $\phi^*\colon \mathbb{C}[W] \to \mathbb{C}[V]$. The important property of this functor is, that a morphism of affine varieties $\phi\colon V \to W$ is an isomorphism if and only if $\phi^*\colon \mathbb{C}[W] \to \mathbb{C}[V]$ is an isomorphism, see [CLO07, Chapter 5, §4, Theorem 9]. Thus, we are able to reconstruct an affine variety V from its coordinate ring $\mathbb{C}[V]$ up to isomorphism.

The Spectrum of a $\mathbb{C}$-Algebra. Every coordinate ring is a finitely generated $\mathbb{C}$-algebra with no non-zero nilpotents, since it is a quotient of $\mathbb{C}[x_1, \ldots, x_n]$ by a radical ideal (*i.e.*, an ideal with $\sqrt{I} = I$). On the other hand, given a finitely generated $\mathbb{C}$-algebra R with no non-zero nilpotents, we can always construct an affine variety V such that $\mathbb{C}[V] \cong R$.

Let R be a finitely generated $\mathbb{C}$-algebra with no non-zero nilpotents. Pick generators $f_1, \ldots, f_r \in R$ and define a $\mathbb{C}$-algebra homomorphism $\varphi\colon \mathbb{C}[x_1, \ldots, x_r] \to R$ by $x_i \mapsto f_i$. Since φ is surjective, we have $R \cong \mathbb{C}[x_1, \ldots, x_r]/\ker\varphi$, where $\ker\varphi$ is radical since R has no non-zero nilpotents. Thus, $\mathbf{V}(\ker\varphi) \subseteq \mathbb{C}^r$ is an affine variety with coordinate ring isomorphic to R.

The affine variety determined by a $\mathbb{C}$-algebra R is called $\operatorname{Spec}(R)$. The reason for this notation is that the set of maximal ideals of a ring is called its maximal spectrum. The theory of schemes introduces a more general notion of varieties that would allow us to directly define $\operatorname{Spec}(R)$ as a scheme, without the need to embed it in $\mathbb{C}^n$ like we did in the previous construction. While it is calming to know that we could describe $\operatorname{Spec}(R)$ without involving an arbitrary choice of generators of R, it is enough for our purposes to set $\operatorname{Spec}(R) = \mathbf{V}(\ker\varphi)$ for a fixed choice of generators.

The Complex Torus. The multiplicative group $(\mathbb{C}^*)^n$ is called the n-dimensional *complex torus*. Our first step in the direction of toric geometry is to equip the complex torus with the structure of an affine variety. Strictly speaking, $(\mathbb{C}^*)^n$ is not an affine variety according to Definition 1.1, since it cannot be expressed as the zero-locus of a finite family of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. It rather is the complement of an affine variety, namely $(\mathbb{C}^*)^n = \mathbb{C}^n \setminus \mathbf{V}(x_1x_2\cdots x_n)$. This allows us to construct an affine variety in $\mathbb{C}^n \times \mathbb{C}$ that projects to $(\mathbb{C}^*)^n$ bijectively, which is a general construction we will revisit when discussing localizations of coordinate rings in Section 1.5.

Let $V = \mathbf{V}(1 - x_1x_2\ldots x_ny) \subseteq \mathbb{C}^n \times \mathbb{C}$, where $x_1, \ldots, x_n$ are the coordinates of $\mathbb{C}^n$ and y is the coordinate for the additional factor $\mathbb{C}$. The projection $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ maps V bijectively onto $(\mathbb{C}^*)^n$, equipping it with the structure of an affine variety. Using this construction, we find the coordinate ring of the complex torus to be

$$\begin{aligned}\mathbb{C}[(\mathbb{C}^*)^n] &= \mathbb{C}[x_1, \ldots, x_n, y]/(1 - x_1x_2\ldots x_ny) = \mathbb{C}[x_1, \ldots, x_n, 1/(x_1x_2\ldots x_n)]\\ &= \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}],\end{aligned}$$

which is the ring of *Laurent polynomials* in n variables over $\mathbb{C}$.

1.2 Convex Polyhedral Cones

The heart of toric geometry lies in the fact, that toric varieties arise from a combinatorial structure called a *fan*. In this section we will discuss *cones*, which are the building blocks of fans and belong to affine toric varieties, which will in turn be the building blocks of toric varieties.

Definition 1.5. A lattice N is a free abelian group of finite rank, *i.e.* $N \cong \mathbb{Z}^n$. It is contained in the real vector space $N_{\mathbb{R}} = N \otimes \mathbb{R} \cong \mathbb{R}^n$. The dual lattice M given as $\operatorname{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$ is contained in $M_{\mathbb{R}} \cong \mathbb{R}^n$, which is the dual vector space to $N_{\mathbb{R}}$.

We have a product $\langle -, - \rangle \colon M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ given by the usual dual pairing of vector spaces. This will be our standard setting in the following sections, so M will always be lattice with dual N and $M_{\mathbb{R}}$, $N_{\mathbb{R}}$ their corresponding ambient real vector spaces.

Definition 1.6. A *convex polyhedral cone* in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \operatorname{Cone}(u_1, \dots, u_k) = \left\{ \sum_{i=1}^{k} r_i u_i \;\middle|\; r_i \geq 0 \right\} \subseteq N_{\mathbb{R}},$$

for some $u_1, \dots, u_k \in N_{\mathbb{R}}$. A convex polyhedral cone is called *rational* if all $u_i \in N$.

Since "convex polyhedral cone" is a rather long term, we will usually use the word "cone" and imply that we are talking about convex polyhedral cones.

Dual Cones. Given a cone $\sigma \subseteq N_{\mathbb{R}}$, we define the convex set

$$\sigma^{\vee} = \{ v \in M_{\mathbb{R}} \mid \langle v, u \rangle \geq 0 \text{ for all } u \in \sigma \}.$$

This set is called the *dual cone* of σ, which is justified by the following proposition.

Proposition 1.7 ([Ewa96, Chapter V Theorem 2.1, Lemma 2.2, Theorem 2.9])**.** *Given a convex polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, its dual cone $\sigma^{\vee} \subseteq M_{\mathbb{R}}$ is again a convex polyhedral cone. We have $(\sigma^{\vee})^{\vee} = \sigma$ and $\sigma^{\vee}$ is rational if and only if σ is rational.* □

Example 1.8. Consider the lattice $N = \mathbb{Z}^2$ with ambient real vector space $N_\mathbb{R} = \mathbb{R}^2$. The rational cone $\sigma = \mathrm{Cone}(2e_1 + e_2\,, e_2) \subseteq \mathbb{R}^2$ is the intersection of two half-spaces given by the inward pointing normal vectors e_1^* and $-e_1^* + 2e_2^*$. Thus, the dual cone is obtained as $\sigma^\vee = \mathrm{Cone}(e_1^*\,, -e_1^* + 2e_2^*) \subseteq M_\mathbb{R} = \mathbb{R}^2$, as illustrated in Figure 1.2. $\diamondsuit$

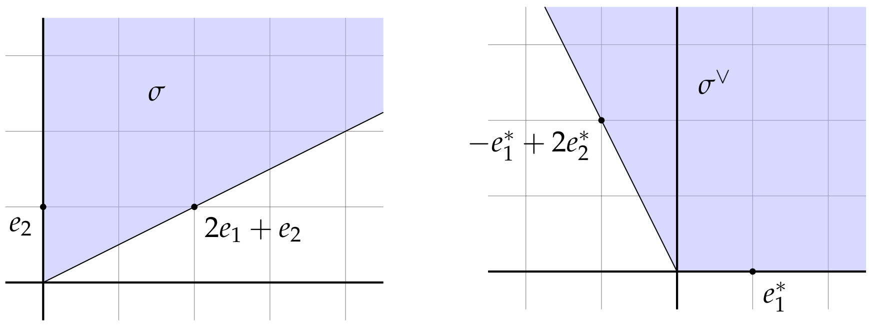


Figure 1.2: The cone $\sigma = \mathrm{Cone}(2e_1 + e_2\,, e_2) \subseteq \mathbb{R}^2$ and its dual.

Using a description of σ as an intersection of half-spaces as we did in Example 1.8 is a general construction to obtain the dual cone $\sigma^\vee$.

1.3 Semigroups and Semigroup Algebras

To obtain an affine variety from a cone, we will first construct a $\mathbb{C}$-algebra from the cone, that will serve as a coordinate ring. These algebras will be given by semigroups.

Definition 1.9. A *semigroup* is a subset S of a lattice M that is closed under addition and contains 0. The semigroup S is said to be *generated by* a subset $A \subseteq S$, if

$$S = \mathbb{N}A = \left\{ \sum_{a\in A} k_a a \;\middle|\; k_a \in \mathbb{N}, \text{only finitely many } k_a \neq 0 \right\}.$$

The semigroup S is *finitely generated* if there is some finite set A generating S.

Proposition 1.10 (Gordan's Lemma). *If $\sigma \subseteq N_\mathbb{R}$ is a rational convex polyhedral cone, then $S_\sigma = \sigma^\vee \cap M$ is a finitely generated semigroup.*

Proof. The subset $S_\sigma = \sigma^\vee \cap M$ of M is a semigroup, since both $\sigma^\vee$ and M contain 0 and are closed under addition. Since σ is a rational cone, we know that $\sigma^\vee$ is a rational cone as well by Proposition 1.7. Thus $\sigma^\vee = \operatorname{Cone}(v_1, \dots, v_s)$ for some $v_i \in M$. Now consider the set

$$K = \left\{ \sum_{i=1}^{s} t_i v_i \,\middle|\, t_i \in [0,1] \right\} \subseteq \sigma^\vee.$$

Since K is bounded, $K \cap M$ is finite. We will show that S_σ is generated by $K \cap M$.

Take any $v \in S_\sigma = \sigma^\vee \cap M$, then $v = \sum_{i=1}^{s} r_i v_i$ for some $r_i \geq 0$. We have

$$v = \sum_{i=1}^{s} \lfloor r_i \rfloor v_i + \sum_{i=1}^{s} (r_i - \lfloor r_i \rfloor) v_i,$$

where $\lfloor r \rfloor$ denotes the integer part of a non-negative real number r.

Since v and the first summand are elements of M, the second summand is in M as well. Since $v_i \in K \cap M$, the first summand is in $\mathbb{N}(K \cap M)$. The second summand is in K since $r_i - \lfloor r_i \rfloor \leq 1$, thus $v \in \mathbb{N}(K \cap M)$. □

Example 1.8 (continuing from p. 12)**.** We obtain a generating set of $S_\sigma = \sigma^\vee \cap \mathbb{Z}^2$ as $K \cap M = \{0, e_1^*, e_2^*, 2e_2^*, -e_1^* + 2e_2^*\}$, as shown in Figure 1.3. Since $2e_2^*$ is generated by e_2^*, it can be omitted, hence $S_\sigma = \mathbb{N}\{e_1^*, e_2^*, -e_1^* + 2e_2^*\}$. ◊

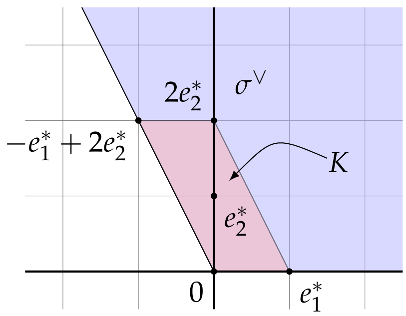


Figure 1.3: The generating set of the semigroup $S_\sigma = \sigma^\vee \cap M$.

Remark 1.11. At this point it is unclear, why we are considering $\sigma^\vee \cap M$ instead of $\sigma \cap N$. After all, $\sigma \cap N$ is also a finitely generated semigroup that seems to be more closely related to the cone σ. The reason for this is, that we want the faces of σ to correspond to

Zariski open subsets of the associated affine variety, so we can glue the cones of a fan along these open subsets. In Proposition 1.22 we will see how this works out in detail.

Semigroup Algebras. Now we associate $\mathbb{C}$-algebras to semigroups, which will then allow us to define affine toric varieties.

Definition 1.12. Let S be a semigroup in the lattice M. The *semigroup algebra* $\mathbb{C}[S]$ is given as a vector space with basis elements χ^m for all $m \in S$. The multiplication in $\mathbb{C}[S]$ is defined by $\chi^m \chi^{m'} = \chi^{m+m'}$. If S is generated by $m_1, \ldots, m_s$, we write $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}]$, where the relations are implicit in the definition of the multiplication.

Example 1.8 (continuing from p. 12)**.** In the second part of Example 1.8 we constructed the semigroup $S_\sigma = \mathbb{N}\{e_1^*, e_2^*, -e_1^* + 2e_2^*\}$. To this semigroup, we associate the $\mathbb{C}$-algebra $\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}, \chi^{-e_1^*+2e_2^*}]$ which is isomorphic to $\mathbb{C}[x, y, x^{-1}y^2]$ by mapping $x \mapsto \chi^{e_1^*}$ and $y \mapsto \chi^{e_2^*}$. ♢

1.4 Affine Toric Varieties

Definition 1.13. An affine variety V is *toric*, if $V = \operatorname{Spec}(\mathbb{C}[S])$ for some finitely generated semigroup S. If $V = \operatorname{Spec}(\mathbb{C}[S_\sigma])$ for a rational cone σ, we write $V = U_\sigma$.

Remark 1.14. An affine toric variety $V = \operatorname{Spec}(\mathbb{C}[S])$ always contains a torus $T \cong (\mathbb{C}^*)^k$ as a Zariski open subset. We will come back to this fact when discussing how localizations of $\mathbb{C}[S]$ correspond to Zariski open subsets of V. The characters $\chi\colon T \to \mathbb{C}$ form a lattice isomorphic to $\mathbb{Z}S$. In fact, we can define affine toric varieties using embedded tori and their actions on V (see [CLS11, Definition 1.1.3]), which is the historic approach that led to toric geometry.

We should note that not every affine toric variety comes from a cone. For example the semigroup $\{0, 2, 3, 4, \ldots\} \subseteq \mathbb{Z}$ can not come from a cone, since it contains 2 but not 1, which is contained in $\operatorname{Cone}(2) = \mathbb{R}_{\geq 0}$. Semigroups S that contain m, whenever $km \in S$ for some positive integer k are called *saturated* and those are exactly the semigroups that arise from cones. It turns out that an affine toric variety $V = \operatorname{Spec}(\mathbb{C}[S])$ is *normal* if and only if S is saturated, so it comes from a cone.

Example 1.8 (continuing from p. 12)**.** From $\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}, \chi^{-e_1^*+2e_2^*}]$ we have

$$\mathbb{C}[S_\sigma] \cong \mathbb{C}[x, y, x^{-1}y^2] \cong \mathbb{C}[x, y, z]/\langle xz - y^2\rangle,$$

so $U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$ is isomorphic to the affine variety $\mathbf{V}(xz - y^2)$ in $\mathbb{C}^3$. We observe that the relation $xz = y^2$ corresponds to the linear relation $e_1^* + (-e_1^* + 2e_2^*) = 2e_2^*$ between the generators of S_σ. ◊

The previous observation generalizes to all affine toric varieties. For $V = \operatorname{Spec}(\mathbb{C}[S])$, we can describe the ideal I such that $V = \mathbf{V}(I)$ in terms of the linear relations between the generators of S.

Proposition 1.15. *Let $S \subseteq M$ be a semigroup with generators $A = \{m_1 \dots, m_s\}$, then* $\operatorname{Spec}(\mathbb{C}[S]) = \mathbf{V}(I) \subseteq \mathbb{C}^s$ *for the ideal*

$$I = \left\langle x^a - x^b \,\middle|\, a, b \in \mathbb{N}^s \text{ such that } \sum_{i=1}^{s} a_i m_i = \sum_{i=1}^{s} b_i m_i \right\rangle, \quad \text{where } x^a = x_1^{a_2} x_2^{a_2} \cdots x_s^{a_s}.$$

Proof. By our previous construction, we have $\operatorname{Spec}(\mathbb{C}[S]) = \mathbf{V}(\ker\varphi) \subseteq \mathbb{C}^s$ for the homomorphism $\varphi\colon \mathbb{C}[x_1, \dots, x_s] \to \mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}]$, given by $x_i \mapsto \chi^{m_i}$. Let $x^a - x^b \in I$, then

$$\varphi(x^a - x^b) = (\chi^{m_1})^{a_1} \cdots (\chi^{m_s})^{a_s} - (\chi^{m_1})^{b_1} \cdots (\chi^{m_s})^{b_s} = \chi^{\sum_{i=1}^{s} a_i m_i} - \chi^{\sum_{i=1}^{s} b_i m_i} = 0,$$

so we have $I \subseteq \ker\varphi$.

Now let $f = \sum c_a x^a \in \ker\varphi$ and define for any $m \in S$ the set $\pi(m)$ of multi-indices $a \in \mathbb{N}^s$ such that $\sum_{i=1}^{s} a_i m_i = m$. We have

$$\varphi(f) = \sum_{m \in S} \left(\sum_{a \in \pi(m)} c_a \right) \chi^m = 0,$$

and therefore $\sum_{a\in\pi(m)} c_a = 0$ for all $m \in S$. It suffices to show that $f_m = \sum_{a\in\pi(m)} c_a x^a$ lies in the ideal I for all $m \in S$. Let $c_{a^1}, \dots, c_{a^k}$ be the non-zero coefficients in f_m, then

$$f_m = \sum_{i=1}^{k} c_{a^i} x^{a^i} = c_{a^1}\left(x^{a^1} - x^{a^2}\right) + (c_{a^2} + c_{a^1})\left(x^{a^2} - x^{a^3}\right) + (c_{a^3} + c_{a^2} + c_{a^1})\left(x^{a^3} - x^{a^4}\right) + \cdots + \left(\sum_{i=1}^{k} c_{a^i}\right)\left(x^{a^k} - x^{a^1}\right) + \left(\sum_{i=1}^{k} c_{a^i}\right) x^{a^1}.$$

The last term vanishes, since $\sum_{i=1}^{k} c_{a^i} = 0$, and all other terms are elements of I, so we have $\ker\varphi \subseteq I$. $\square$

Remark 1.16. Prime ideals generated by binomials are called *toric ideals*. We have seen in Proposition 1.15 that every affine toric variety is given by a toric ideal: The ideal I is evidently generated by binomials. To see that I is prime, note that $\mathbb{C}[M]$ is the ring of Laurent polynomials in the χ^{e_i}, hence an integral domain. So $\mathbb{C}[S] \subseteq \mathbb{C}[M]$ is an integral domain as well and since $\mathbb{C}[S] \cong \mathbb{C}[x_1, \dots, x_s]/I$, we see that I is prime. In fact, $\mathbf{V}(I)$ is an affine toric variety if and only if I is a toric ideal. For a proof see [CLS11, Theorem 1.1.17].

Our goal is to understand how affine toric varieties corresponding to cones in a fan are glued together to a toric variety. Now that we understand how cones relate to affine toric varieties, our next step is to understand how faces of cones correspond to certain Zariski open subsets of the varieties.

1.5 Localizations of Coordinate Rings

Consider an affine variety $V \subseteq \mathbb{C}^n$ with coordinate ring $\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_n]/\mathbf{I}(V)$. Assuming V is irreducible (*i.e.*, $\mathbb{C}[V]$ is an integral domain, so it has a field of fractions $\mathbb{C}(V)$) we define the *localization* at $f \in \mathbb{C}[V] \setminus \{0\}$ by

$$\mathbb{C}[V]_f = \left\{ \frac{g}{f^k} \in \mathbb{C}(V) \,\middle|\, g \in \mathbb{C}[V], k \geq 0 \right\} = \mathbb{C}[V][1/f].$$

Proposition 1.17. *Let $V \subseteq \mathbb{C}^n$ be an irreducible affine variety, $f \in \mathbb{C}[V] \setminus \{0\}$, then*

$$\operatorname{Spec}(\mathbb{C}[V]_f) = V_f := \{\, p \in V \mid f(p) \neq 0 \,\}.$$

Proof. As in our discussion of the complex torus, the Zariski open set V_f is not an affine variety a priori. Though, we can use the same construction to find an affine variety $W \subseteq \mathbb{C}^n \times \mathbb{C}$ that projects bijectively onto V_f. Let $\mathbf{I}(V) = \langle f_1, \dots, f_s \rangle$ and define $W = \mathbf{V}(f_1, \dots, f_s, 1 - gy)$, where $g \in \mathbb{C}[x_1, \dots, x_n]$ represents $f \in \mathbb{C}[V]$. The projection $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ maps W bijectively onto V_f, equipping it with the structure of an affine variety. Using this identification we obtain the coordinate ring

$$\begin{aligned}\mathbb{C}[V_f] &= \mathbb{C}[W] = \mathbb{C}[x_1, \dots, x_n, y]/\langle f_1, \dots, f_s, 1 - gy \rangle \\ &= \mathbb{C}[x_1, \dots, x_n, 1/g]/\langle f_1, \dots, f_s \rangle = \mathbb{C}[V][1/f] = \mathbb{C}[V]_f. \end{aligned} \qquad \square$$

Remark 1.18. Given a finitely generated semigroup algebra $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}]$, we have

$$\mathbb{C}[S]_{\chi^{m_1} \cdots \chi^{m_s}} = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}, \chi^{-m_1 - \dots - m_s}] = \mathbb{C}[\chi^{\pm m_1}, \dots, \chi^{\pm m_s}] = \mathbb{C}[\mathbb{Z}S],$$

so $\operatorname{Spec}(\mathbb{C}[\mathbb{Z}S])$ is a Zariski open subset of the affine toric variety $\operatorname{Spec}(\mathbb{C}[S])$. Since $\mathbb{Z}S \cong \mathbb{Z}^k$ for some $k \in \mathbb{N}$, we know that $\mathbb{C}[\mathbb{Z}S]$ is the ring of Laurent polynomials in k variables, so $\operatorname{Spec}(\mathbb{C}[\mathbb{Z}S]) \cong (\mathbb{C}^*)^k$. This is the torus contained in every affine toric variety, as mentioned in Remark 1.14.

Example 1.8 (continuing from p. 12)**.** We got $U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma]) \cong \mathbf{V}(xz - y^2) \subseteq \mathbb{C}^3$ from the semigroup algebra $\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}, \chi^{-e_1^* + 2e_2^*}]$ of the cone $\sigma = \operatorname{Cone}(2e_1^* + e_2^*, e_2^*)$ in $\mathbb{R}^2$. To find the embedded torus, we look at

$$\mathbb{C}[\mathbb{Z}S_\sigma] = \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}, \chi^{-e_1^* + 2e_2^*}]_{\chi^{3e_2^*}} = \mathbb{C}[S_\sigma]_{\chi^{e_2^*}},$$

so we find $\operatorname{Spec}(\mathbb{C}[\mathbb{Z}S_\sigma]) = (U_\sigma)_{\chi^{e_2^*}} \cong \mathbf{V}(xz - y^2)_y$. All points of $\mathbf{V}(xz - y^2)$ with $y \neq 0$ also have $x, z \neq 0$, since $xz = y^2$ for points on U_σ. Thus we have the torus

$$T = \operatorname{Spec}(\mathbb{C}[\mathbb{Z}S_\sigma]) \cong \left\{ (x, y, x^{-1}y^2) \;\middle|\; x, y \in \mathbb{C}^* \right\} \subseteq \mathbf{V}(xz - y^2) \subseteq \mathbb{C}^3,$$

which is isomorphic to $(\mathbb{C}^*)^2$ by $(x, y) \mapsto (x, y, x^{-1}y^2)$. $\Diamond$

1.6 Faces of Cones and Zariski Open Subsets

Definition 1.19. Let $\sigma \subseteq N_{\mathbb{R}}$ be a convex polyhedral cone. Given $m \in M_{\mathbb{R}}$ we define the hyperplane and half-space

$$H_m = \{\, u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0 \} \subseteq N_{\mathbb{R}},$$
$$H_m^+ = \{\, u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \} \subseteq N_{\mathbb{R}}.$$

If $\sigma \subseteq H_m^+$ we call H_m a *supporting hyperplane* of σ. This happens if and only if $m \in \sigma^\vee$.

Note that we allow $m = 0$, so we have a degenerate supporting hyperplane $H_0 = N_{\mathbb{R}}$.

Definition 1.20. A *face* of a cone σ is a subset given as $\tau = \sigma \cap H_m$ for some supporting hyperplane H_m. In this case we write $\tau \preceq \sigma$.

Proposition 1.21. *If $\tau = \sigma \cap H_m$ is a face of the cone $\sigma = \operatorname{Cone}(u_1, \ldots, u_k)$, we have $\tau = \operatorname{Cone}(u_i : u_i \in H_m)$, so every face of a cone is a cone itself.*

Proof. We have $\operatorname{Cone}(u_i : u_i \in H_m) \subseteq \tau = \sigma \cap H_m$, since H_m is a subspace. Now consider any $u \in \tau$, so $u = \sum_{i=1}^k r_i u_i$ and $\langle m, u \rangle = 0$. Since $m \in \sigma^\vee$, we have $\langle m, u_i \rangle \geq 0$ for all i. Thus,

$$0 = \langle m, u \rangle = \sum_{i=1}^k r_i \langle m, u_i \rangle$$

where all $r_i \geq 0$ and all $\langle m, u_i \rangle \geq 0$. We conclude that $r_i = 0$, whenever $\langle m, u_i \rangle > 0$, which is equivalent to $u_i \notin H_m$. Therefore $u = \sum_{u_i \in H_m} r_i u_i$ as desired. □

If σ is rational, every face $\tau = \sigma \cap H_m$ is given by some $m \in S_\sigma = \sigma^\vee \cap M$, since all we need is that $\langle m, u_i \rangle$ vanishes whenever $u_i \in \tau$ and is strictly positive when $u_i \notin \tau$. Having all u_i rational, we can choose m rational as well.

Proposition 1.22. *Let $\tau = \sigma \cap H_m$ be a face of a rational convex polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ given by $m \in S_\sigma$. Then the affine toric variety U_τ is the Zariski open subset $(U_\sigma)_{\chi^m}$ of U_σ.*

Proof. From $\tau = \sigma \cap H_m$ we obtain the dual cone $\tau^\vee = \operatorname{Cone}(\sigma^\vee \cup \{-m\})$, since adding $-m$ to $\sigma^\vee$ has the effect of intersecting σ with the half-space H_{-m}^+, which is the same as intersecting with H_m since $\sigma \subseteq H_m^+$. Thus, we have $S_\tau = \tau^\vee \cap M = S_\sigma + \mathbb{Z}(-m)$ and the coordinate ring is the localization

$$\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma + \mathbb{Z}(-m)] = \mathbb{C}[S_\sigma][\chi^{-m}] = \mathbb{C}[S_\sigma]_{\chi^m}.$$

Therefore, by Proposition 1.17,

$$U_\tau = \operatorname{Spec}(\mathbb{C}[S_\tau]) = \operatorname{Spec}(\mathbb{C}[S_\sigma]_{\chi^m}) = (U_\sigma)_{\chi^m}. \qquad \square$$

Example 1.23. Let $N = \mathbb{Z}^3$ and consider the cone $\sigma = \operatorname{Cone}(e_1, e_2, e_1+e_3, e_2+e_3)$ in $N_\mathbb{R} = \mathbb{R}^3$. By describing σ as the intersection of four half-spaces given by its facets, we obtain the dual cone $\sigma^\vee = \operatorname{Cone}(e_1^*, e_2^*, e_3^*, e_1^*+e_2^*-e_3^*) \subseteq \mathbb{R}^3$, as shown in Figure 1.4.

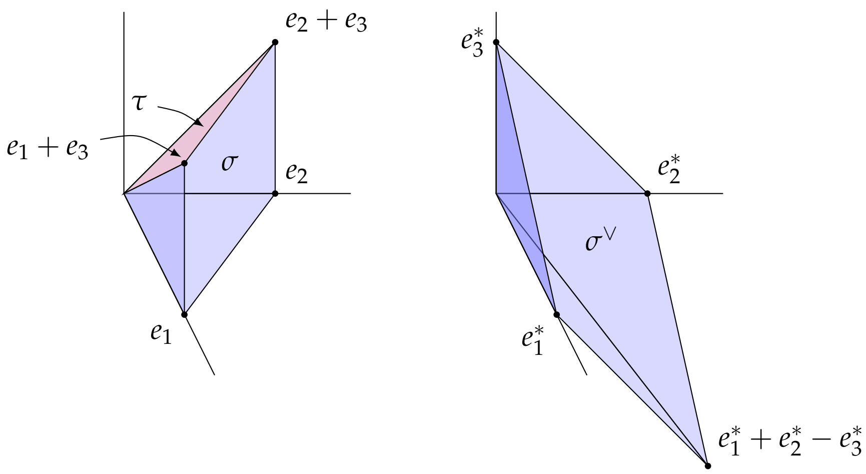


Figure 1.4: The cone $\sigma = \operatorname{Cone}(e_1, e_2, e_1+e_3, e_2+e_3) \subseteq \mathbb{R}^3$ and its dual.

The semigroup $S_\sigma = \sigma^\vee \cap \mathbb{Z}^3$ is generated by e_1^*, e_2^*, e_3^* and $e_1^*+e_2^*-e_3^*$, so we obtain the semigroup algebra

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}, \chi^{e_3^*}, \chi^{e_1^*+e_2^*-e_3^*}] \cong \mathbb{C}[x, y, z, xyz^{-1}] \cong \mathbb{C}[x, y, z, w]/\langle xy - zw\rangle.$$

We conclude that $U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$ is isomorphic to $\mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$.

The face $\tau = \operatorname{Cone}(e_1+e_3, e_2+e_3)$ (marked red in Figure 1.4) is obtained as $\tau = \sigma \cap H_m$ for $m = e_1^*+e_2^*-e_3^* \in S_\sigma$, so the associated affine toric variety U_τ is the Zariski open subset $(U_\sigma)_{\chi^{e_1^*+e_2^*-e_3^*}}$, which is isomorphic to

$$\mathbf{V}(xy-zw)_w = \left\{ (x,y,z,w) \in \mathbb{C}^4 \,\middle|\, xy = zw, w \neq 0 \right\}. \qquad \Diamond$$

Strong Convexity. We know from Remark 1.18 that a toric variety $\operatorname{Spec}(\mathbb{C}[S])$ always contains the torus $\operatorname{Spec}(\mathbb{C}[\mathbb{Z}S])$ as a Zariski open subset. The dimension of this torus is given by the rank of $\mathbb{Z}S$, which might be different from the rank of the lattice M containing S in general. If we want the contained torus to have the dimension given by the rank of M, we need $\mathbb{Z}S = M$. For arbitrary finitely generated semigroups $S \subseteq M$ this condition is equivalent to S containing a basis of M. However, if $S = S_\sigma$ is given by a rational cone $\sigma \subseteq N_\mathbb{R}$, this condition is equivalent to σ being *strongly convex*, which means that $\{0\}$ is a face of σ or equivalently, σ contains no positive dimensional subspace of $N_\mathbb{R}$.

Proposition 1.24. *Let $\sigma \subseteq N_\mathbb{R} \cong \mathbb{R}^n$ be a rational convex polyhedral cone with semigroup $S_\sigma = \sigma^\vee \cap M$. Then $\mathbb{Z}S_\sigma = M$ if and only if σ is a strongly convex cone. In this case the torus of U_σ has dimension n, the rank of the lattices N and M.*

Proof. Let $\sigma \subseteq N_\mathbb{R}$ be strongly convex, then $\tau = \{0\}$ is a face of σ, so $\tau = \sigma \cap H_m$ for some $m \in S_\sigma$. Since $\tau^\vee = M_\mathbb{R}$ we have $S_\sigma + \mathbb{Z}(-m) = S_\tau = \tau^\vee \cap M = M$, so $\mathbb{Z}S_\sigma = M$. Conversely, if $\mathbb{Z}S_\sigma = M$, we know that S_σ contains a basis $m_1, \ldots, m_n$ of M. Let $m = m_1 + \cdots + m_n$, then $\sigma \cap H_m = \{0\}$, since any $u \in \sigma \cap H_m$ satisfies $0 = \langle m, u \rangle = \sum_{i=1}^n \langle m_i, u \rangle$, where all $\langle m_i, u \rangle \geq 0$, so in fact all $\langle m_i, u \rangle = 0$ and thus $u = 0$ since $m_1, \ldots, m_n$ is a basis of $M_\mathbb{R}$. $\square$

For a strongly convex rational polyhedral cone $\sigma \subseteq N_\mathbb{R}$, the edges are always *rays*, since σ contains no 1-dimensional subspace. Along each ray $\rho \preceq \sigma$, there is a unique $u_\rho \in \rho \cap N$ that generates the semigroup $\rho \cap N$. The collection of all u_ρ, where ρ ranges over the edges of σ, is called the collection of *minimal ray generators* of σ. In fact, $\sigma = \operatorname{Cone}(u_{\rho_1}, \ldots, u_{\rho_r})$, so the minimal ray generators always generate σ as a cone (see [CLS11, Lemma 1.2.15]).

1.7 Toric Varieties from Polyhedral Fans

In order to glue affine toric varieties along the Zariski open subsets corresponding to faces of the underlying cones, we need a general construction of gluing affine varieties along Zariski open subsets.

Definition 1.25. An *abstract variety* is given by a finite family $(V_\alpha)_{\alpha \in I}$ of affine varieties, Zariski open sets $V_{\beta\alpha} \subseteq V_\alpha$ for all pairs $\alpha, \beta \in I$ and isomorphisms $g_{\beta\alpha} \colon V_{\beta\alpha} \to V_{\alpha\beta}$, satisfying the following conditions:

(a) For every pair $\alpha, \beta \in I$ the isomorphisms $g_{\alpha\beta}$ and $g_{\beta\alpha}$ are mutually inverse.

(b) For all $\alpha, \beta, \gamma \in I$ we have $g_{\beta\alpha}(V_{\beta\alpha} \cap V_{\gamma\alpha}) = V_{\alpha\beta} \cap V_{\gamma\beta}$ and $g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha}$ on $V_{\beta\alpha} \cap V_{\gamma\alpha}$.

These conditions give an equivalence relation on the disjoint union $\coprod_{\alpha \in I} V_\alpha$ by letting $a \sim b$ if and only if $a \in V_{\beta\alpha}$, $b \in V_{\alpha\beta}$ and $g_{\beta\alpha}(a) = b$ for some $\alpha, \beta \in I$. The abstract variety given by this data is the quotient space

$$X = \coprod_{\alpha \in I} V_\alpha \Big/ \sim.$$

Abstract varieties have a standard and a Zariski topology, obtained by equipping each V_α with the standard or Zariski topology, respectively. The images of the affine varieties V_α in the quotient X are called the *affine charts* of the abstract variety X.

Remark 1.26. In order to determine if two abstract varieties are isomorphic, we would need some definition of morphisms between abstract varieties. Defining those morphisms properly involves *rings of regular functions* and *sheaves*, that describe what kind of maps correspond to our polynomial maps in the affine setting, where the coordinate ring encoded this information. For a proper definition see [CLS11, § 3.0]. Since we are more concerned with topological features like cohomology in this thesis, we skip this definition and instead give correspondences of affine charts whenever we mention an isomorphism of abstract varieties.

Example 1.27. Let V_1 and V_2 be two copies of $\mathbb{C}$ and define the Zariski open subsets $V_{21} = \mathbb{C}^* \subseteq V_1$, $V_{12} = \mathbb{C}^* \subseteq V_2$. Consider the isomorphisms $\mathbb{C}^* \to \mathbb{C}^*$ given by $g : z \mapsto z$

and $\widetilde{g} : z \mapsto 1/z$. Gluing V_1 and V_2 along g, we obtain the abstract variety X_g, which might be described as the complex line with two origins, since all other points of the two copies have been identified. Gluing along $\widetilde{g}$, we obtain a different abstract variety $X_{\widetilde{g}}$, which is isomorphic to $\mathbb{CP}^1$, since the gluing exactly mimics how the two charts $\{[1 : z] \mid z \in \mathbb{C}\}$ and $\{[z : 1] \mid z \in \mathbb{C}\}$ intersect in $\mathbb{CP}^1$. $\Diamond$

Remark 1.28. Some authors call the object defined in Definition 1.25 a *prevariety* and require abstract varieties to be *separated*, which is equivalent to being Hausdorff with respect to the standard topology. In this terminology, the complex line with two origins is a non-separated prevariety. We will not make this distinction, since all toric varieties obtained from polyhedral fans are separated (see [CLS11, Theorem 3.1.5]).

Since faces of cones correspond to Zariski open subsets of the associated affine toric varieties, we can glue affine toric varieties U_{σ_1} and U_{σ_2} that are given by cones σ_1 and σ_2 intersecting in a common face $\sigma_1 \cap \sigma_2$ along $U_{\sigma_1 \cap \sigma_2}$. The structure needed to obtain a toric variety in this way is a fan.

Definition 1.29. A *rational polyhedral fan* (or just *fan*) Σ in $N_{\mathbb{R}}$ for a lattice $N \cong \mathbb{Z}^n$ is a finite collection of cones $\sigma \subseteq N_{\mathbb{R}}$ satisfying the following conditions:

(a) Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.

(b) For $\sigma \in \Sigma$ and $\tau \preceq \sigma$, we have $\tau \in \Sigma$.

(c) For $\sigma_1, \sigma_2 \in \Sigma$, we have $\sigma_1 \cap \sigma_2 \preceq \sigma_1, \sigma_2$.

The k-dimensional cones in Σ form a subset $\Sigma^{(k)} \subseteq \Sigma$. In particular, since every cone is strongly convex, $\Sigma^{(1)}$ is a set of rays. The fan Σ is called *complete*, if every $u \in N_{\mathbb{R}}$ is contained in some $\sigma \in \Sigma$.

Definition 1.30. Given a rational polyhedral fan Σ, the family of affine toric varieties $(U_\sigma)_{\sigma \in \Sigma}$ and Zariski open subsets $U_{\sigma_2,\sigma_1} = U_{\sigma_1 \cap \sigma_2} \subseteq U_{\sigma_1}$ with the obvious isomorphisms $U_{\sigma_1,\sigma_2} \cong U_{\sigma_2,\sigma_1}$ define an abstract variety called the *toric variety* X_Σ.

The condition of strong convexity in Definition 1.29 guarantees that all of the glued affine toric varieties U_σ contain the same torus $U_{\{0\}} = \operatorname{Spec}(\mathbb{C}[M])$, which is identified to a single torus in X_Σ. The other two conditions establish the gluing conditions needed to construct an abstract variety.

Remark 1.31. As in the case of affine toric varieties in Remark 1.14, toric varieties can also be defined using embedded tori and their actions on the abstract variety (see [CLS11, Definition 3.1.1]). Similar to the affine case, toric varieties defined this way will not always come from a fan. Again the toric varieties obtained from rational polyhedral fans are exactly the *normal* toric varieties.

Example 1.32. Let $N = \mathbb{Z}^2$ and consider the fan Σ given by the cones $\sigma_1 = \operatorname{Cone}(e_1, e_2)$, $\sigma_2 = \operatorname{Cone}(e_1, -e_1 - e_2)$, $\sigma_3 = \operatorname{Cone}(e_2, -e_1 - e_2)$ and all of their faces. Describing the cones as intersections of half-spaces we obtain the dual cones from the inward pointing normal vectors as $\sigma_1^\vee = \operatorname{Cone}(e_1^*, e_2^*)$, $\sigma_2^\vee = \operatorname{Cone}(e_1^* - e_2^*, -e_2^*)$ and $\sigma_3^\vee = \operatorname{Cone}(e_2^* - e_1^*, -e_1^*)$, as illustrated in Figure 1.5.

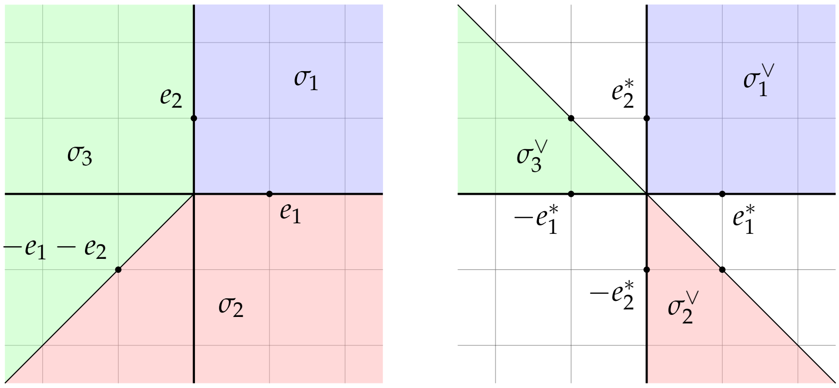


Figure 1.5: The fan Σ of $\mathbb{CP}^2$ and the duals of its maximal cones.

Note that we only need to glue U_{σ_1}, U_{σ_2} and U_{σ_3} along their common Zariski open subsets to obtain X_Σ, since all other cones in Σ are faces of these three cones and the corresponding affine toric varieties will be glued in as already existing Zariski open subsets.

Let us calculate the three semigroup algebras to obtain the affine charts of X_Σ.

$$\begin{aligned}
\mathbb{C}[S_{\sigma_1}] &= \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}] \cong \mathbb{C}[x_1, y_1] && \Longrightarrow && U_{\sigma_1} = \operatorname{Spec}(\mathbb{C}[S_{\sigma_1}]) \cong \mathbb{C}^2, \\
\mathbb{C}[S_{\sigma_2}] &= \mathbb{C}[\chi^{e_1^*-e_2^*}, \chi^{-e_2^*}] \cong \mathbb{C}[x_2, y_2] && \Longrightarrow && U_{\sigma_2} = \operatorname{Spec}(\mathbb{C}[S_{\sigma_2}]) \cong \mathbb{C}^2, \\
\mathbb{C}[S_{\sigma_3}] &= \mathbb{C}[\chi^{e_2^*-e_1^*}, \chi^{-e_1^*}] \cong \mathbb{C}[x_3, y_3] && \Longrightarrow && U_{\sigma_3} = \operatorname{Spec}(\mathbb{C}[S_{\sigma_3}]) \cong \mathbb{C}^2.
\end{aligned}$$

For each cone, the two generators already generate the semigroup and since they form a basis of M in each case, there are no linear relations, so the ideal from Proposition 1.15 is trivial. We chose $\mathbb{C}$-algebra isomorphism given by $x_1, y_1 \mapsto \chi^{e_1^*}, \chi^{e_2^*}$ for $\mathbb{C}[S_{\sigma_1}]$, $x_2, y_2 \mapsto \chi^{e_1^*-e_2^*}, \chi^{-e_2^*}$ for $\mathbb{C}[S_{\sigma_2}]$ and $x_3, y_3 \mapsto \chi^{e_2^*-e_1^*}, \chi^{-e_1^*}$ for $\mathbb{C}[S_{\sigma_3}]$ to keep track of the coordinates in the three copies of $\mathbb{C}^2$.

The affine charts U_{σ_1} and U_{σ_2} are glued along $U_{\sigma_1 \cap \sigma_2}$, where $\sigma_1 \cap \sigma_2 = \mathrm{Cone}(e_1)$ with dual $(\sigma_1 \cap \sigma_2)^\vee = \mathrm{Cone}(e_1^*, e_2^*, -e_2^*)$ and semigroup algebra $\mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}, \chi^{-e_2^*}]$. Expressing this semigroup algebra as a localization of the semigroup algebras of σ_1 and σ_2 using $\sigma_1 \cap \sigma_2 = \sigma_1 \cap H_{e_2^*} = \sigma_2 \cap H_{-e_2^*}$, we find the Zariski open subsets that need to be glued.

$$\begin{aligned}
\mathbb{C}[S_{\sigma_1 \cap \sigma_2}] &= \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}, \chi^{-e_2^*}] \\
&= \mathbb{C}[S_{\sigma_1}]_{\chi^{e_2^*}} \cong \mathbb{C}[x_1, y_1]_{y_1} \\
&= \mathbb{C}[S_{\sigma_2}]_{\chi^{-e_2^*}} \cong \mathbb{C}[x_2, y_2]_{y_2}.
\end{aligned}$$

Thus, the Zariski open subset is $\mathbb{C} \times \mathbb{C}^*$ given by $y_1 \neq 0$ for U_{σ_1} and $y_2 \neq 0$ for U_{σ_2}. We need to identify $(x_1, y_1) \in \mathbb{C}^2 \cong U_{\sigma_1}$ with $(x_2, y_2) \in \mathbb{C}^2 \cong U_{\sigma_2}$ whenever $y_1, y_2 \neq 0$ and $x_2 = x_1 y_1^{-1}, y_2 = y_1^{-1}$, obtained from $\chi^{e_1^*-e_2^*} = \chi^{e_1^*}(\chi^{e_2^*})^{-1}$ and $\chi^{-e_2^*} = (\chi^{e_2^*})^{-1}$ under the chosen isomorphisms. The gluing rules for $U_{\sigma_1}, U_{\sigma_3}$ and $U_{\sigma_2}, U_{\sigma_3}$ are obtained similarly and reveal that X_Σ is isomorphic to $\mathbb{CP}^2$, where the three affine charts of X_Σ correspond to the three charts $(x_1 : y_1 : 1)$, $(x_2 : 1 : y_2)$ and $(1 : x_3 : y_3)$ in $\mathbb{CP}^2$. $\diamondsuit$

2 Combinatorial Chow Rings

In this chapter we define and study combinatorial Chow rings of simplicial fans. As we will see in Chapter 3, these rings appear as cohomology rings of the associated toric varieties in some cases. The goal of this chapter is to understand the linear structure of the combinatorial Chow ring of a fan Σ without referring to the associated toric variety X_Σ. Hence, we restrict ourselves to algebraic and combinatorial tools to obtain results depending only on the immediate features of the fan Σ.

2.1 Abstract Simplicial Complexes of Simplicial Fans

We have seen in Proposition 1.21, that every face of a cone is given by a subset of its generators. We now define a class of cones where the converse, Proposition 2.2, holds as well.

Definition 2.1. Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone. The cone σ is *simplicial* if its minimal ray generators are linearly independent over $\mathbb{R}$. If the minimal ray generators form part of a $\mathbb{Z}$-basis of M, we say σ is *smooth* or *unimodular*. In particular, every smooth cone is simplicial.

Proposition 2.2. *Let $\sigma = \operatorname{Cone}(u_1, \dots, u_k) \subseteq N_\mathbb{R}$ be a simplicial cone, then the cones $\operatorname{Cone}(R)$ for $R \subseteq \{u_1, \dots, u_k\}$ are the faces of σ.*

Proof. Let $\tau = \sigma \cap H_m$ be a face of σ, given by some $m \in \sigma^\vee$. By Proposition 1.21 we know that $\tau = \operatorname{Cone}(\, u_i : u_i \in H_m \,)$. Thus, all faces of σ are given as $\operatorname{Cone}(R)$ for some $R \subseteq \{u_1, \dots, u_k\}$.

Since $u_1, \dots, u_k$ are linearly independent, we can extend to a basis $u_1, \dots, u_n$ of $N_{\mathbb{R}}$ with the dual basis $u_1^*, \dots, u_n^*$ of $M_{\mathbb{R}}$. Now given any subset $R \subseteq \{u_1, \dots, u_k\}$, let $m = \sum_{u_i \notin R} u_i^*$. From Proposition 1.21 we have $\sigma \cap H_m = \operatorname{Cone}(u_i : i \leq k, u_i \in H_m)$, where $u_i \in H_m$ is equivalent to $i \in R$ by the choice of m and $\langle u_i^*, u_j \rangle = \delta_{ij}$. Thus $\operatorname{Cone}(R) = \sigma \cap H_m$ is a face of σ. □

The notions of being simplicial or smooth can be extended to fans.

Definition 2.3. A fan Σ in $N_{\mathbb{R}}$ is *simplicial* if all of its cones are simplicial. The fan is *smooth* or *unimodular*, if all of its cones are smooth.

Since the faces of simplicial cones are given by subsets of its minimal ray generators, we can associate an abstract simplicial complex to a simplicial fan. Before we do that, we give a proper definition of abstract simplicial complexes and some of their properties.

Definition 2.4. An *abstract simplicial complex* Δ on a finite vertex set V is a collection of subsets of V, such that $A \subseteq B \in \Delta$ implies $A \in \Delta$. The elements of Δ are called *faces* and the *dimension* of a face $F \in \Delta$ is $\dim F = |F| - 1$. The dimension of a non-empty abstract simplicial complex Δ is $\dim \Delta = \max_{F \in \Delta} \dim F$. The inclusionwise maximal faces are called *facets* and Δ is called *d-pure* if all facets are of equal dimension d.

Proposition 2.5. *Let Σ be a simplicial fan in $N_{\mathbb{R}}$ with rays $\Sigma^{(1)} = \{\rho_1, \dots, \rho_r\}$ and minimal ray generators $V_\Sigma = \{u_1, \dots, u_r\} \subseteq N$, then*

$$\Delta(\Sigma) = \{R \subseteq V_\Sigma \,|\, \operatorname{Cone}(R) \in \Sigma\}$$

is an abstract simplicial complex with vertices V_Σ and faces corresponding to the cones in Σ.

Proof. By definition, the elements of $\Delta(\Sigma)$ are subsets of V_Σ. Now let $R \in \Delta(\Sigma)$, then $\operatorname{Cone}(R)$ is a simplicial cone in Σ, thus by Proposition 2.2 all $\operatorname{Cone}(R')$ for $R' \subseteq R$ are faces of $\operatorname{Cone}(R)$, hence in the fan Σ, so $R' \in \Delta(\Sigma)$. □

2.2 Combinatorial Chow Rings

Now that we understand the structure of simplicial fans, we can define the combinatorial Chow ring. The definition we give is the same as Ewald's definition of the combinatorial Chow ring in [Ewa96, Chapter VII Definition 5.1] for complete smooth fans, leaving out the assumption of completeness and weakening smoothness to simpliciality.

Definition 2.6. Let Σ be a simplicial fan in $N_{\mathbb{R}}$. Fix a numbering $\rho_1, \ldots, \rho_r$ of the rays in $\Sigma^{(1)}$ with minimal ray generators $u_1, \ldots, u_r \in N$. For every ray ρ_i introduce a formal variable X_i. For every cone $\sigma = \operatorname{Cone}(u_{i_1}, \ldots, u_{i_s})$ with $1 \le i_1 < \cdots < i_s \le r$ define the square-free monomial $P_\sigma = X_{i_1} \cdots X_{i_s} \in \mathbb{Z}[X_1, \ldots, X_r]$. In this polynomial ring, we define the ideals

$$\begin{aligned} \mathcal{I} &= \langle\, P_\sigma \,|\, \sigma \notin \Sigma \,\rangle\,, \\ \mathcal{J} &= \langle\, \langle m, u_1\rangle X_1 + \cdots + \langle m, u_r\rangle X_r \,|\, m \in M \,\rangle\,. \end{aligned}$$

The *combinatorial Chow ring* of Σ is defined as $\mathcal{R}(\Sigma) = \mathbb{Z}[X_1, \ldots, X_r]/(\mathcal{I} + \mathcal{J})$. For convenience, we define $x_i = [X_i] \in \mathcal{R}(\Sigma)$ for $i = 1, \ldots, r$ and $p_\sigma = [P_\sigma] \in \mathcal{R}(\Sigma)$. Considering $\mathcal{I}$ and $\mathcal{J}$ as ideals in $\mathbb{Q}[X_1, \ldots, X_r]$ we define the *rational combinatorial Chow ring* of Σ as $\mathcal{R}_{\mathbb{Q}}(\Sigma) = \mathbb{Q}[X_1, \ldots, X_r]/(\mathcal{I} + \mathcal{J}) = \mathcal{R}(\Sigma) \otimes \mathbb{Q}$.

Remark 2.7. In the definition of $\mathcal{I}$ it is enough to consider all *minimal non-cones* of Σ, *i.e.* cones $\operatorname{Cone}(R) \notin \Sigma$ for $R \subseteq \{u_1, \ldots, u_r\}$ such that $\operatorname{Cone}(R') \in \Sigma$ for all $R' \subsetneq R$. In the definition of $\mathcal{J}$ it is enough to choose a basis $m_1, \ldots, m_n$ of M and let $\mathcal{J}$ be generated by the $\sum_{i=1}^{r} \langle m_j, u_i\rangle X_i$ for $j = 1, \ldots, n$.

Remark 2.8. The combinatorial Chow ring of a simplicial fan Σ is closely related to the *Stanley-Reisner ring* or *face ring* of the associated simplicial complex $\Delta(\Sigma)$, denoted by $k[\Delta(\Sigma)]$, obtained as the quotient of the polynomial ring $k[X_1, \ldots, X_r]$ over a field k, by the *Stanley-Reisner ideal* $\mathcal{I}$ defined as in Definition 2.6. We see that the ideal $\mathcal{I}$ only depends on the combinatorial structure of Σ that is captured in $\Delta(\Sigma)$, while the additional ideal $\mathcal{J}$ encodes information of the coordinates of the ray generators. Stanley-Reisner rings of abstract simplicial complexes have been studied in detail, see [Sta96, Chapter II].

A very useful lemma for studying the combinatorial Chow ring is the following *shifting lemma*. It is a generalization of [Ewa96, Chapter VII Lemma 5.3] to non-complete fans and at the same time slightly stronger by allowing only σ_j above τ.

Lemma 2.9. *Let Σ be a smooth fan in $N_{\mathbb{R}}$. If $\tau \prec \sigma \preceq \sigma' \in \Sigma$, then there exist cones $\sigma_j \in \Sigma$ with $\dim \sigma_j = \dim \sigma$ and integers c_j for $j = 1, \ldots, q$, such that $\tau \prec \sigma_j \npreceq \sigma'$ and*

$$p_\sigma = c_1 p_{\sigma_1} + \cdots + c_q p_{\sigma_q} \in \mathcal{R}(\Sigma).$$

Proof. Let n be the rank of M. Fix a numbering of the minimal ray generators of Σ such that $\sigma' = \mathrm{Cone}(u_1, \ldots, u_d)$, $\sigma = \mathrm{Cone}(u_1, \ldots, u_s)$ and $\tau = \mathrm{Cone}(u_k, \ldots, u_s)$ for $1 < s \leq d \leq n$ and $0 \leq s - k + 1 < s$, so τ is a proper face of σ, allowing $\tau = \{0\}$ when $k = s + 1$. Since Σ is smooth, we can extend $u_1, \ldots, u_d$ to a $\mathbb{Z}$-basis $v_1, \ldots, v_n$ of N, where $v_i = u_i$ for $i = 1, \ldots, d$. This yields a dual basis $v_1^*, \ldots, v_n^*$ of M. In particular, for $m = v_1^*$ we obtain

$$\langle v_1^*, u_1 \rangle x_1 + \cdots + \langle v_1^*, u_r \rangle x_r = 0.$$

Since $v_i = u_i$ for $i = 1, \ldots, d$ we have $\langle v_1^*, u_1 \rangle = 1$ and $\langle v_1^*, u_i \rangle = 0$ for $i = 2, \ldots, d$. Hence,

$$x_1 = -\langle v_1^*, u_{d+1} \rangle x_{d+1} - \cdots - \langle v_1^*, u_r \rangle x_r.$$

Substituting into p_σ gives

$$\begin{aligned} p_\sigma = x_1 \cdots x_s &= (-\langle v_1^*, u_{d+1} \rangle x_{d+1} - \cdots - \langle v_1^*, u_r \rangle x_r) x_2 \cdots x_s \\ &= c_{d+1} x_{d+1} x_2 \cdots x_s + \cdots + c_r x_r x_2 \cdots x_s \\ &= c_{d+1} p_{\sigma_{d+1}} + \cdots + c_r p_{\sigma_r}, \end{aligned}$$

with $c_i = -\langle v_1^*, u_i \rangle$ and $\sigma_i = \mathrm{Cone}(u_i, u_2, \ldots, u_s)$ for $i = d + 1, \ldots, r$. Since every ray generator of τ is contained in σ_i, we have $\tau \prec \sigma_i$ for $i = d + 1, \ldots, r$. For $\sigma_i \notin \Sigma$, we have $p_{\sigma_i} = 0$, so the corresponding term vanishes. For $\sigma_i \in \Sigma$, we have $\sigma_i \npreceq \sigma'$, since ρ_i is a ray of σ_i but not a ray of σ'. □

Remark 2.10. If Σ is only simplicial, we obtain the same result for $\mathcal{R}_{\mathbb{Q}}(\Sigma)$: From an integral basis $v_1, \ldots, v_n$ of $N_{\mathbb{R}}$ we get a rational dual basis $v_1^*, \ldots, v_n^*$ of $M_{\mathbb{R}}$ that can be transformed into an integral basis of $M_{\mathbb{R}}$ by scaling. As a result, the coefficients c_i are no longer integral, but still rational.

The first step in the direction of a linear basis of $\mathcal{R}(\Sigma)$ is the following theorem, telling us that the combinatorial Chow ring is linearly generated by square-free monomials. This theorem also appears as [Ewa96, Chapter IV Theorem 5.5]. The proof given by Ewald holds in the non-complete case as well.

Theorem 2.11. *Let Σ be a smooth fan in $N_{\mathbb{R}} \cong \mathbb{R}^n$ and $\mathcal{R}^{(s)}(\Sigma)$ be the subgroup of $\mathcal{R}(\Sigma)$ generated by the square-free monomials of degree s. Then $\mathcal{R}(\Sigma)$ decomposes as a graded ring*

$$\mathcal{R}(\Sigma) = \mathcal{R}^{(0)}(\Sigma) \oplus \cdots \oplus \mathcal{R}^{(n)}(\Sigma).$$

Proof. We start by showing that every monomial $x_{i_1}^{r_1} \cdots x_{i_t}^{r_t} \in \mathcal{R}(\Sigma)$ can be expressed as a linear combination of square-free monomials of the same degree. If the largest exponent is 1, the monomial is already square-free. Otherwise, without loss of generality, $r_1 > 1$ is the largest exponent. If $\sigma = \mathrm{Cone}(u_{i_1}, \ldots, u_{i_t}) \notin \Sigma$, the monomial is zero. If $\sigma \in \Sigma$, we apply Lemma 2.9 for $\{0\} \prec \rho_{i_1} \preceq \sigma$ to obtain

$$\begin{aligned} x_{i_1}^{r_1} \cdots x_{i_t}^{r_t} &= x_{i_1} x_{i_1}^{r_1-1} \cdots x_{i_t}^{r_t} = (c_1 x_{j_1} + \cdots + c_q x_{j_q}) x_{i_1}^{r_1-1} \cdots x_{i_t}^{r_t} \\ &= c_1 x_{j_1} x_{i_1}^{r_1-1} \cdots x_{i_t}^{r_t} + \cdots + c_q x_{j_q} x_{i_1}^{r_1-1} \cdots x_{i_t}^{r_t}, \end{aligned}$$

where all $\rho_{j_k} \not\preceq \sigma$, so $x_{j_k} \notin \{x_{i_1}, \ldots, x_{i_t}\}$. Applying the Lemma for all x_{i_k} with $r_k = r_1$, we reduce the largest exponent to $r_1 - 1$. By induction, we obtain an expression of $x_{i_1}^{r_1} \cdots x_{i_t}^{r_t}$ as a linear combination of square-free monomials of the same degree. Since $\mathcal{I}$ and $\mathcal{J}$ are homogeneous ideals, the standard grading of $\mathbb{Z}[X_1, \ldots, X_r]$ induces the desired grading on $\mathcal{R}(\Sigma)$. $\square$

Remark 2.12. If Σ is only simplicial, the same decomposition into subgroups generated by square-free monomials of the same degree holds for $\mathcal{R}_{\mathbb{Q}}(\Sigma)$, by applying the simplicial version of Lemma 2.9 mentioned in Remark 2.10.

2.3 Products of Fans

At this point, we make a digression from our way to a linear basis of $\mathcal{R}(\Sigma)$ to study combinatorial Chow rings of products of fans. The connections between combinatorial Chow rings and Stanley-Reisner rings and between products of simplicial fans and joins of abstract simplicial complexes suggest, how the combinatorial Chow ring should behave under taking products. We use this as a sanity check for our general definition of combinatorial Chow rings.

Definition 2.13. Let Σ_1, Σ_2 be fans in $(N_1)_\mathbb{R}$ and $(N_2)_\mathbb{R}$, respectively. The *product fan* $\Sigma_1 \times \Sigma_2$ is the fan in $(N_1)_\mathbb{R} \times (N_2)_\mathbb{R} = (N_1 \times N_2)_\mathbb{R}$ with cones $\sigma_1 \times \sigma_2$ for $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$.

Assuming Σ_1 and Σ_2 are simplicial, we can translate the product construction to the associated abstract simplicial complexes. We see that

$$\Delta(\Sigma_1 \times \Sigma_2) = \{\, R \cup S \subseteq V_{\Sigma_1} \cup V_{\Sigma_2} \mid R \in \Delta(\Sigma_1), S \in \Delta(\Sigma_2) \,\} = \Delta(\Sigma_1) * \Delta(\Sigma_2),$$

which is the *join* of the abstract simplicial complexes $\Delta(\Sigma_1)$ and $\Delta(\Sigma_2)$. For Stanley-Reisner rings it holds that $k[\Delta_1 * \Delta_2] = k[\Delta_1] \otimes_k k[\Delta_2]$, thus it is to be expected that $\mathcal{R}(\Sigma_1 \times \Sigma_2)$ behaves similarly.

Proposition 2.14. *Let Σ_1, Σ_2 be simplicial fans in $(N_1)_\mathbb{R}$ and $(N_2)_\mathbb{R}$. There is a natural ring isomorphism $\mathcal{R}(\Sigma_1 \times \Sigma_2) \cong \mathcal{R}(\Sigma_1) \otimes_\mathbb{Z} \mathcal{R}(\Sigma_2)$.*

Proof. Let $\rho_1, \ldots, \rho_{r_1}$ be the rays of Σ_1 with minimal generators $u_1, \ldots, u_{r_1}$ and $\tau_1, \ldots, \tau_{r_2}$ the rays of Σ_2 with minimal generators $v_1, \ldots, v_{r_2}$. We have

$$\begin{aligned} \mathcal{R}(\Sigma_1) &= \mathbb{Z}[X_1, \ldots, X_{r_1}]/(\mathcal{I}_1 + \mathcal{J}_1), \\ \mathcal{R}(\Sigma_2) &= \mathbb{Z}[Y_1, \ldots, Y_{r_2}]/(\mathcal{I}_2 + \mathcal{J}_2). \end{aligned}$$

The tensor product $\mathcal{R}(\Sigma_1) \otimes_\mathbb{Z} \mathcal{R}(\Sigma_2)$ is naturally isomorphic to the quotient of the polynomial ring $\mathbb{Z}[X_1, \ldots, X_{r_1}, Y_1, \ldots, Y_{r_2}]$ by the corresponding ideal extension $\mathcal{I}_1 + \mathcal{J}_1 + \mathcal{I}_2 + \mathcal{J}_2$.

The rays of $\Sigma_1 \times \Sigma_2$ are $\rho_i \times \{0\}$ for $i = 1, \ldots, r_1$ and $\{0\} \times \tau_j$ for $j = 1, \ldots, r_2$, so

$$\mathcal{R}(\Sigma_1 \times \Sigma_2) = \mathbb{Z}[X_1, \ldots, X_{r_1}, Y_1, \ldots, Y_{r_2}]/(\mathcal{I} + \mathcal{J}).$$

Thus, all we need to verify is $\mathcal{I} + \mathcal{J} = \mathcal{I}_1 + \mathcal{J}_1 + \mathcal{I}_2 + \mathcal{J}_2$. Let $P_{\sigma_1 \times \sigma_2} \in \mathcal{I}$, so that $\sigma_1 \times \sigma_2 \notin \Sigma_1 \times \Sigma_2$. This only happens when $\sigma_1 \notin \Sigma_1$ or $\sigma_2 \notin \Sigma_2$. Without loss of generality, assume $\sigma_1 \notin \Sigma_1$ so that $P_{\sigma_1} \in \mathcal{I}_1$. Since $P_{\sigma_1 \times \sigma_2} = P_{\sigma_1} P_{\sigma_2}$, it follows that $P_{\sigma_1 \times \sigma_2} \in \mathcal{I}_1$. Let $(m_1, m_2) \in M_1 \times M_2$ give a generator of $\mathcal{J}$. We have

$$\sum_{i=1}^{r_1} \Big\langle (m_1, m_2), (u_1, 0) \Big\rangle X_i + \sum_{j=1}^{r_2} \Big\langle (m_1, m_2), (0, u_2) \Big\rangle Y_j = \underbrace{\sum_{i=1}^{r_1} \langle m_1, u_1 \rangle X_i}_{\in \mathcal{J}_1} + \underbrace{\sum_{j=1}^{r_2} \langle m_2, u_2 \rangle Y_r}_{\in \mathcal{J}_2} .$$

Therefore, we have shown that $\mathcal{I} + \mathcal{J} \subseteq \mathcal{I}_1 + \mathcal{J}_1 + \mathcal{I}_2 + \mathcal{J}_2$. For the other inclusion, we notice that $P_{\sigma_1} = P_{\sigma_1 \times \{0\}}$, so that $\mathcal{I}_1 \subseteq \mathcal{I}$ since $\sigma_1 \notin \Sigma_1$ implies $\sigma_1 \times \{0\} \notin \Sigma_1 \times \Sigma_2$. By the same argument $\mathcal{I}_2 \subseteq \mathcal{I}$. From the equation above we see that $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{J}$ by letting $m_2 = 0$ and $m_1 = 0$, respectively. $\square$

2.4 Shellability

The property of Σ that will make our construction of a linear basis for $\mathcal{R}(\Sigma)$ work is the notion of *shellability*. We define shellability of abstract simplicial complexes as in [BW96] and note an important lemma that characterizes shellability by a way of partitioning the complex.

For any sets $A \subseteq B$ in an abstract simplicial complex, the *Boolean interval from A to B* is the set of all C such that $A \subseteq C \subseteq B$, denoted $[A, B]$. For any set A we define $\overline{A} = [\varnothing, A]$.

Definition 2.15. An abstract simplicial complex Δ is *shellable* if its facets can be arranged in a linear order $F_1, F_2, \ldots F_t$, such that the subcomplex $\left(\bigcup_{i=1}^{k-1} \overline{F}_i \right) \cap \overline{F}_k$ is pure of dimension $\dim F_k - 1$ for all $k = 2, \ldots, t$. An ordering of the facets satisfying this condition is called a *shelling* or *shelling order* of Δ.

The *restriction map* $R : \{F_1, F_2, \ldots, F_n\} \to \Delta$ is defined by

$$R(F_k) = \left\{ v \in F_k \;\middle|\; F_k \setminus \{v\} \in \bigcup_{i=1}^{k-1} \overline{F}_i \right\}.$$

Björner and Wachs showed that shellability is equivalent to being able to partition the complex into Boolean intervals, where the partition has to satisfy an additional ordering condition. Abstract simplicial complex that allow a partition into Boolean intervals are called *partitionable*. Thus, every shellable complex is partitionable, but the converse does not hold.

Proposition 2.16 ([BW96, Proposition 2.5]). *Let $F_1, F_2, \ldots, F_t$ be an ordering of the facets of an abstract simplicial complex Δ and $R : \{F_1, F_2, \ldots, F_t\} \to \Delta$ a map. Then $F_1, F_2, \ldots, F_t$ is a shelling with restriction map R if and only if $\Delta = \dot\bigcup_{i=1}^{t}[R(F_i), F_i]$ and $R(F_i) \subseteq F_j$ implies $i \leq j$ for all i, j.*

We will refer to the partition $\Delta = \dot\bigcup_{i=1}^{t}[R(F_i), F_i]$ induced by a shelling of Δ with restriction map R as a *shelling partition*.

The notion of shellability can be directly transferred to simplicial fans.

Definition 2.17. A simplicial fan Σ in $N_{\mathbb{R}}$ is *shellable* if the associated abstract simplicial complex $\Delta(\Sigma)$ is shellable. An ordering of the maximal cones of Σ inducing a shelling of $\Delta(\Sigma)$ is a *shelling* of Σ.

2.5 Linear Generators in the Shellable Case

We now know all the properties of the combinatorial Chow ring and shellability that allow us to formulate a linear generating set of $\mathcal{R}(\Sigma)$ that is the candidate for the linear basis we are looking for.

Theorem 2.18. *Let Σ be a smooth shellable fan in $N_{\mathbb{R}}$ with shelling order $\sigma_1, \ldots, \sigma_t$. Then the monomials $p_{R(\sigma_i)}$ belonging to the restrictions of the σ_i generate $\mathcal{R}(\Sigma)$ as an abelian group.*

Proof. Let $\sigma_1 \ldots, \sigma_t$ be a shelling order of Σ. By Proposition 2.16 we have

$$\Sigma = \dot\bigcup_{i=1}^{t} [R(\sigma_i), \sigma_i], \quad \text{such that } R(\sigma_i) \preceq \sigma_j \text{ implies } i \leq j,$$

where $[\tau, \sigma]$ denotes the set of all cones σ' with $\tau \preceq \sigma' \preceq \sigma$.

We will use backwards induction on the shelling order to show that every p_σ with $\sigma \in [R(\sigma_i), \sigma_i]$ can be expressed as a linear combination of the restrictions $p_{R(\sigma_j)}$ for $j \geq i$.

For $i = t$ consider $R(\sigma_t) \prec \sigma \preceq \sigma_t$. Applying Lemma 2.9 we have

$$p_\sigma = c_1 p_{\tilde{\sigma}_1} + \cdots + c_q p_{\tilde{\sigma}_q},$$

where $R(\sigma_t) \prec \tilde{\sigma}_j \npreceq \sigma_t$ for $j = 1, \ldots, q$. Since $[R(\sigma_t), \sigma_t]$ is the last interval in the shelling of Σ, there are no cones above $R(\sigma_t)$, that aren't faces of σ_t, thus $p_\sigma = 0$.

For $i < l$ and $R(\sigma_i) \prec \sigma \preceq \sigma_i$, we apply the shifting lemma to obtain p_σ as a linear combination of some $p_{\tilde{\sigma}_j}$ with $R(\sigma_i) \prec \tilde{\sigma}_j \npreceq \sigma_i$. Every $\tilde{\sigma}_j$ is contained in an interval $[R(\sigma_k), \sigma_k]$ for some $k > i$, since $R(\sigma_i) \prec \tilde{\sigma}_j \preceq \sigma_k$ and $\tilde{\sigma}_j \notin [R(\sigma_i), \sigma_i]$. Thus, by induction, all $p_{\tilde{\sigma}_j}$ are linear combinations of monomials given by restrictions.

After t steps we reach $i = 1$ which finishes the proof, since all intervals have been covered. □

Example 2.19. Consider the fan Σ in $\mathbb{R}^2$ with $X_\Sigma \cong \mathbb{CP}^2$ from Example 1.32. The ray generators of Σ are $u_1 = e_1$, $u_2 = e_2$ and $u_3 = -e_1 - e_2$. Identifying cones in Σ with subsets of $\{1, 2, 3\}$, we obtain the face poset of Σ as shown in Figure 2.1.

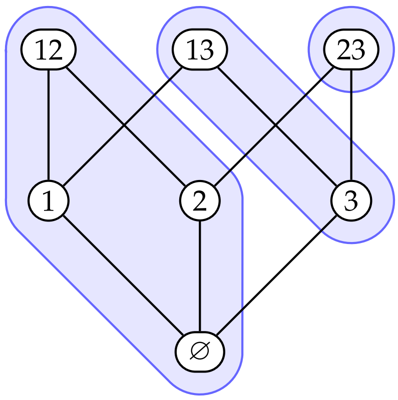


Figure 2.1: The face poset of Σ for $X_\Sigma \cong \mathbb{CP}^2$ with shelling partition.

The combinatorial Chow ring of Σ is obtained as

$$\mathcal{R}(\Sigma) = \mathbb{Z}[X_1, X_2, X_3] / \langle X_1 X_2 X_3\,,\ X_2 - X_3\,,\ X_1 - X_3 \rangle \cong \mathbb{Z}[X] / \langle X^3 \rangle.$$

As seen in Figure 2.1, the fan is shellable with shelling order 12, 13, 23. The restrictions are $R(12) = \varnothing$, $R(13) = 3$ and $R(23) = 23$ with corresponding monomials $p_\varnothing = 1$,

$p_3 = x$ and $p_{23} = x^2$. We see that $(1, x, x^2)$ linearly generate $\mathcal{R}(\Sigma)$. In fact, we found a linear basis of the combinatorial Chow ring. It is no coincidence that $\mathbb{Z}[X]/\langle X^3 \rangle$ is also the cohomology ring $H^*(\mathbb{CP}^2)$. We will come back to this connection in Chapter 3. $\Diamond$

Up to dimension 1 we can prove that, as long as Σ is n-pure, the linear generators given by Theorem 2.18 are linearly independent, as noticed in Example 2.19.

Proposition 2.20. *The monomials $p_{R(\sigma_i)}$ belonging to the s-dimensional restrictions of the cones in a shelling order of a smooth, n-pure fan Σ in $N_{\mathbb{R}} \cong \mathbb{R}^n$ form a linear basis of $\mathcal{R}^{(s)}(\Sigma)$ for $s = 0, 1$.*

Proof. The only cone of dimension 0 is $\{0\} = R(\sigma_1)$. We have $p_{\{0\}} = 1 \neq 0$ and $k \cdot 1 \neq 0$ in $\mathcal{R}(\Sigma)$ for all $k \in \mathbb{Z}$, since there are no relations in degree 0. Hence, $\mathcal{R}^{(0)}(\Sigma) = \mathbb{Z}$ generated by $p_{R(\sigma_1)}$.

For $s = 1$ order the rays so that $\sigma_1 = \mathrm{Cone}(u_1, \dots, u_n)$, since Σ is n-pure. Then $\rho_{n+1}, \dots, \rho_r$ are all the restrictions of dimension 1. Every linear relation in $\mathcal{R}(\Sigma)$ is of the form

$$\langle m, u_1 \rangle x_1 + \cdots + \langle m, u_r \rangle x_r = 0$$

for some $m \in M$, since sums and integral multiples of the relations in $\mathcal{J}$ are obtained by sums and integral multiples of the corresponding $m \in M$. Thus, if there is a linear relation involving only $x_{n+1}, \dots, x_r$, it is given by an $m \in M$ such that $\langle m, u_i \rangle = 0$ for $i = 1, \dots, n$. Since σ_1 is smooth, the $u_1, \dots, u_n$ form a $\mathbb{Z}$-basis of N, so $m = 0$ and the linear combination was trivial. $\square$

Remark 2.21. Again, the results of Theorem 2.18 and Proposition 2.20 still hold for $\mathcal{R}_{\mathbb{Q}}(\Sigma)$ when Σ is only simplicial.

We strongly believe that this linear independence holds in higher dimensions as well, assuming Σ is a smooth, shellable, n-pure fan. In fact, we know it does when Σ is complete, as will be discussed in Section 3.2. From the algebraic point of view we took in this chapter, we formulate the following conjecture.

Conjecture 2.22. *Let Σ be a smooth, shellable, n-pure fan in $N_{\mathbb{R}}$ with shelling order $\sigma_1, \dots, \sigma_t$. Then the monomials $p_{R(\sigma_i)}$ belonging to the restrictions of the σ_i form a linear basis of $\mathcal{R}(\Sigma)$. The same holds for $\mathcal{R}_{\mathbb{Q}}(\Sigma)$ when Σ is only simplicial.*

We finish the chapter by discussing two more examples: A simplicial shellable fan, that is neither smooth nor complete to check the linear basis for $\mathcal{R}_{\mathbb{Q}}(\Sigma)$, and a smooth non-shellable fan, where we can still find a basis of the combinatorial Chow ring by explicit computation.

Example 2.23. Let $N = \mathbb{Z}^2$ and Σ be the fan in $\mathbb{R}^2$ given by the three maximal cones $\sigma_1 = \mathrm{Cone}(e_1, e_2)$, $\sigma_2 = \mathrm{Cone}(e_2, e_2 - e_1)$, $\sigma_3 = \mathrm{Cone}(e_2 - e_1, -e_1 - e_2)$, as shown in Figure 2.2. Note that σ_3 is not smooth, since $e_2 - e_1$, $-e_1 - e_2$ do not generate M.

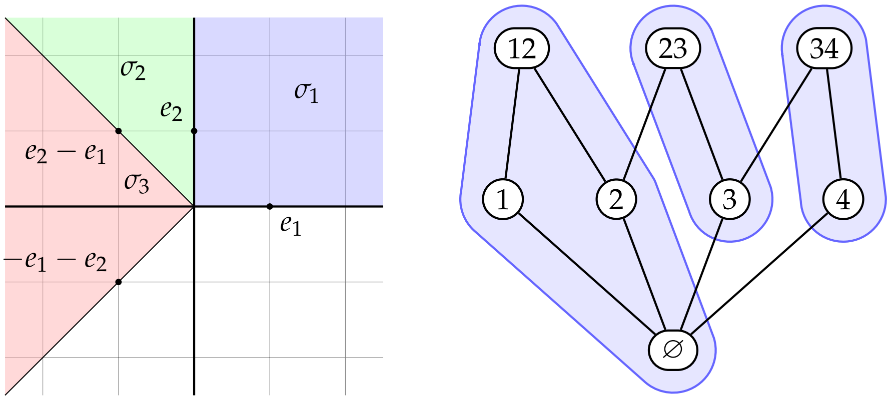


Figure 2.2: The non-smooth, non-complete, simplicial shellable fan Σ and its face poset.

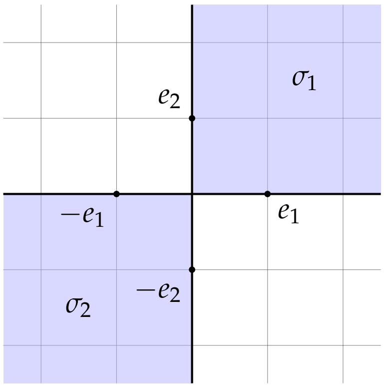
Identifying cones in Σ with subsets of $\{1, 2, 3, 4\}$ corresponding to the rays given by e_1, e_2, $e_2 - e_1$ and $-e_1 - e_2$, we find the ordering $\sigma_1 = 12$, $\sigma_2 = 23$, $\sigma_3 = 34$ is a shelling order with restrictions $\varnothing$, 3, 4, as seen in Figure 2.2.

The combinatorial Chow ring is obtained as

$$\begin{aligned}
\mathcal{R}(\Sigma) &= \frac{\mathbb{Z}[X_1, X_2, X_3, X_4]}{\langle X_1X_3\,,\, X_1X_4\,,\, X_2X_4\,,\, X_1 - X_3 - X_4\,, X_2 + X_3 - X_4 \rangle} \\
&\cong \frac{\mathbb{Z}[X_3, X_4]}{\langle {X_3}^2 + X_3X_4\,,\, X_3X_4 + {X_4}^2\,,\, {X_4}^2 - X_3X_4 \rangle} \\
&\cong \frac{\mathbb{Z}[X_3, X_4]}{\langle {X_3}^2 - {X_4}^2\,,\, 2{X_4}^2\,,\, X_4^2 - X_3X_4 \rangle}.
\end{aligned}$$

We see that ${x_3}^2 = {x_4}^2 = x_3x_4 \neq 0$, but $2{x_4}^2 = 0$ in $\mathcal{R}(\Sigma)$. In particular, the monomials 1, x_3 and x_4 belonging to the restrictions of the shelling do not generate $\mathcal{R}(\Sigma)$ linearly, since $\mathcal{R}^{(2)}(\Sigma) \cong \mathbb{Z}_2 \neq 0$. However, in $\mathcal{R}_{\mathbb{Q}}(\Sigma)$ we get $x_4{}^2 = 0$ after division by 2, hence all degree 2 monomials vanish and $\mathcal{R}_{\mathbb{Q}}(\Sigma) = \langle 1, x_3, x_4 \rangle$ has the desired linear basis. $\Diamond$

Example 2.24. Let $N = \mathbb{Z}^2$ and Σ be the smooth fan in $\mathbb{R}^2$ given by the maximal cones $\sigma_1 = \text{Cone}(e_1, e_2)$, $\sigma_2 = \text{Cone}(-e_1, -e_2)$, as shown in Figure 2.3, together with the non-shellable poset obtained by identifying cones in Σ with subsets of $\{1,2,3,4\}$ corresponding to the rays given by e_1, e_2, $-e_1$ and $-e_2$.



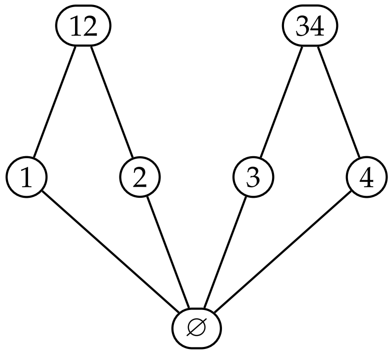


Figure 2.3: The non-shellable, non-complete, smooth fan Σ and its face poset.

The combinatorial Chow ring is obtained as

$$\begin{aligned}\mathcal{R}(\Sigma) &= \frac{\mathbb{Z}[X_1, X_2, X_3, X_4]}{\langle X_1X_3,\ X_1X_4,\ X_2X_3,\ X_2X_4,\ X_1 - X_3,\ X_2 - X_4 \rangle} \\ &\cong \frac{\mathbb{Z}[X_1, X_2]}{\langle X_1{}^2,\ X_1X_2,\ X_2{}^2 \rangle},\end{aligned}$$

so all monomials of degree 2 vanish and the monomials 1, x_1 and x_2 form a linear basis of $\mathcal{R}(\Sigma)$. Note that we need three generators, despite the fact that Σ has only two maximal cones.

As discussed in [BM98], the face poset of Σ is the minimal example of a non-shellable poset. It is a “witness to non-shellability” in the sense that it is contained in every non-shellable poset as an induced subposet. See [Wac97] for this and a more general discussion of obstructions to shellability. $\Diamond$

3 Context in Toric Geometry

In this final chapter we want to go back to toric geometry and discuss the connection of our results on the combinatorial Chow ring $\mathcal{R}(\Sigma)$ from Chapter 2 to topological features of the associated toric variety X_Σ.

3.1 Fulton's Condition is Shellability

In [Ful93, Section 5.2] Fulton establishes a linear basis for the homology groups of a complete smooth toric variety $X_\triangle$ whose underlying fan $\triangle$ satisfies a certain combinatorial condition.

> For any ordering $\sigma_1, \dots, \sigma_m$ of the top-dimensional cones, define a sequence of subcones $\tau_i \subset \sigma_i$, $1 \leq i \leq m$, by letting τ_i be the intersection of σ_i with all those σ_j that come after σ_i (*i.e.*, with $j > i$) and that meet σ_i in a cone of dimension $n-1$. [...] In particular, $\tau_1 = \{0\}$, and $\tau_m = \sigma_m$. The key assumption that will make this work is:
>
> $$\text{If } \tau_i \text{ is contained in } \sigma_j \text{, then } i \leq j. \tag{$*$}$$

We notice that this condition is half of what we need to identify the ordering as a shelling with restriction map $R(\sigma_i) = \tau_i$ by Proposition 2.16. In fact, Fulton goes on and proves the following lemma from ($*$).

> **Lemma.** *(a) For each cone γ in $\triangle$ there is a unique $i = i(\gamma)$ such that $\tau_i \subset \gamma \subset \sigma_i$. In fact, $i(\gamma)$ is the smallest integer i such that σ_i contains γ.*
>
> *(b) If γ is a face of γ', then $i(\gamma) \leq i(\gamma')$.*

This is exactly $\triangle = \dot\bigcup_{i=1}^{m} [\tau_i, \sigma_i]$. Thus, Fulton's lemma shows shellability of $\triangle$.

3.2 Homology, Cohomology and Chow Rings

Besides the combinatorial Chow ring discussed in Chapter 2, every algebraic variety X comes with algebro-geometric Chow groups $A_*(X)$. For every $k \geq 0$ the group $A_k(X)$ is defined as the quotient of the free abelian group generated by the k-dimensional irreducible closed subvarieties of X modulo rational equivalence (see [Ful98]). As usual, these groups are put together in an abelian group $A_*(X) = \bigoplus_{k=0}^{\dim X} A_k(X)$.

Reading carefully through the proofs in [Ful93, Section 5.2], we verified that after proving the previous lemma, Fulton never uses ($*$) again. Hence, we can extract the following theorem.

Theorem 3.1 ([Ful93, p. 102, p. 104])**.** *If Σ is a complete smooth shellable fan with shelling order $\sigma_1, \ldots, \sigma_t$, the classes $[V(R(\sigma_i))]$ form a basis for $A_*(X_\Sigma) \cong H_*(X_\Sigma)$. If Σ is only simplicial, the same is true for $A_*(X)_\mathbb{Q} \cong H_*(X_\Sigma; \mathbb{Q})$.*

Here $A_*(X)_\mathbb{Q} = A_*(X) \otimes \mathbb{Q}$ and $V(R(\sigma_i))$ is a subvariety of X_Σ corresponding to the cone $R(\sigma_i)$ called the *orbit closure*. For a discussion on the correspondence between subvarieties of X_Σ and cones in Σ, see the section on the orbit-cone correspondence in [CLS11, § 3.2].

By letting $A^k(X_\Sigma) = A_{n-k}(X_\Sigma)$ we obtain a graded ring $A^*(X_\Sigma) = \bigoplus_k A^k(X)$ equipped with the intersection product, see [Ful93, Section 5.1] or [Dan78, Section 10.7]. Since X_Σ is a smooth orbifold when Σ is complete smooth, we can use Poincaré duality, so $H^{2k}(X_\Sigma) \cong H_{2n-2k}(X_\Sigma)$ as abelian groups. When Σ is only simplicial, X_Σ is still rationally smooth, so Poincaré duality holds over $\mathbb{Q}$, see [CLS11, § 12.4]. Thus, the basis of homology in Theorem 3.1 is also a linear basis of the cohomology ring. This is where the combinatorial Chow ring enters the stage. Danilov proved the following Theorem in [Dan78].

Proposition 3.2 ([Dan78, Theorem 10.8])**.** *If Σ is a complete smooth fan, we have ring isomorphisms $A^*(X_\Sigma) \cong H^*(X_\Sigma) \cong \mathcal{R}(\Sigma)$. If Σ is only simplicial, we have $A^*(X_\Sigma)_\mathbb{Q} \cong H^*(X_\Sigma; \mathbb{Q}) \cong \mathcal{R}_\mathbb{Q}(\Sigma)$.*

This implies that Conjecture 2.22 holds when Σ is assumed to be a *complete* smooth, respectively simplicial, shellable fan. The idea that motivated the conjecture was to prove the basis theorem directly from the combinatorics of Σ, without involving intersection theory on X_Σ. We expect that such an algebraic prove would not depend on Σ being complete, so we weakened the assumption to n-pureness. The fact that this works out in dimension 1 is another hint that the conjecture might be true. Note that even if the conjecture is true, we do not get a cohomology basis for X_Σ, since the isomorphism $H^*(X_\Sigma) \cong \mathcal{R}(\Sigma)$ depends on the completeness of Σ. To stress this point, we come back to the non-complete fan from Example 2.24.

Example 2.24 (continuing from p. 36)**.** The fan Σ in $\mathbb{R}^2$ with maximal cones the two quadrants in Figure 2.3 is smooth, 2-pure but not complete. If the isomorphism from Proposition 3.2 would hold, we would expect $H^3(X_\Sigma) = 0$, since the isomorphism $\mathcal{R}(\Sigma) \xrightarrow{\sim} H^*(X_\Sigma)$ doubles the degree. The two maximal cones of Σ yield two copies of $\mathbb{C}^2$ as affine charts. They are glued along an inclusion of $(\mathbb{C}^*)^2$ that identifies (x, y) in one copy with (x^{-1}, y^{-1}) in the other copy, whenever $x, y \neq 0$. From this description we obtain a Mayer-Vietoris exact sequence for cohomology groups, in particular

$$\cdots \longrightarrow H^2(\mathbb{C}^2) \oplus H^2(\mathbb{C}^2) \longrightarrow H^2((\mathbb{C}^*)^2) \longrightarrow H^3(X_\Sigma) \longrightarrow H^3(\mathbb{C}^2) \oplus H^3(\mathbb{C}^2) \longrightarrow \cdots.$$

Since $\mathbb{C}^2$ is contractible we get an isomorphism $H^3(X_\Sigma) \cong H^2((\mathbb{C}^*)^2)$. Now $(\mathbb{C}^*)^2$ has the homotopy type of the torus $S^1 \times S^1$, so $H^2((\mathbb{C}^*)^2) \cong \mathbb{Z}$. We conclude that $H^3(X_\Sigma)$ is non-zero, so the combinatorial Chow ring is not isomorphic to the cohomology ring. $\diamondsuit$

Bibliography

[BM98] L. Billera and A. Myers. "Shellability of Interval Orders". In: *Order* 15.2 (1998), pp. 113–117. DOI: 10.1023/A:1006196114698.

[BW96] A. Björner and M. Wachs. "Shellable nonpure complexes and posets. I". In: *Transactions of the American Mathematical Society* 348.4 (1996), pp. 1299–1327. DOI: 10.1090/S0002-9947-96-01534-6.

[CK99] D. Cox and S. Katz. *Mirror Symmetry and Algebraic Geometry*. Mathematical surveys and monographs. American Mathematical Society, 1999.

[CLO07] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms*. 3rd ed. Springer, 2007. DOI: 10.1007/978-0-387-35651-8.

[CLS11] D. Cox, J. Little, and H. Schenk. *Toric Varieties*. Graduate Studies in Mathematics 124. American Mathematical Society, 2011.

[Dan78] V. Danilov. "The Geometry of Toric Varieties". In: *Russian Mathematical Surveys* 33.2 (1978), pp. 97–154. DOI: 10.1070/RM1978v033n02ABEH002305.

[Dem70] M. Demazure. "Sous-groupes algébriques de rang maximum du groupe de Cremona". In: *Annales Scientifiques de l'École Normale Supérieure*. 4th ser. 3.4 (1970), pp. 507–588. NUMDAM: ASENS_1970_4_3_4_507_0.

[Ehl75] F. Ehlers. "Eine Klasse komplexer Mannigfaltigkeiten und die Auflösung einiger isolierter Singularitäten". In: *Mathematische Annalen* 218.2 (1975), pp. 127–156. DOI: 10.1007/BF01370816.

[Ewa96] G. Ewald. *Combinatorial Convexity and Algebraic Geometry*. Graduate Texts in Mathematics 168. Springer, 1996. DOI: 10.1007/978-1-4612-4044-0.

[Ful93] W. Fulton. *Introduction to Toric Varieties*. Annals of Mathematics Studies. Princeton University Press, 1993.

Bibliography

[Ful98] W. Fulton. *Intersection Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 1998. DOI: 10.1007/978-1-4612-1700-8.

[KKMS73] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal Embeddings 1*. Lecture Notes in Mathematics 339. Springer, 1973. DOI: 10.1007/BFb0070318.

[MO75] K. Miyake and T. Oda. "Almost homogeneous algebraic varieties under algebraic torus action". In: *Manifolds–Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973)* (1975), pp. 373–381.

[Sta96] R. Stanley. *Combinatorics and Commutative Algebra*. Second Edition. Progress in Mathematics 41. Birkhäuser Boston, 1996. DOI: 10.1007/b139094.

[Wac97] M. Wachs. "Obstructions to Shellability". In: *ArXiv Mathematics e-prints* (1997). arXiv: math/9707216v1.

Declaration

I hereby declare that I produced this thesis without external assistance and that no other than the listed references have been used as sources of information. This thesis has not previously been presented in identical or similar form to any other examination board.

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Bremen, September 8, 2014,